

## Mixed $hp$ -DGFEM for incompressible flows II: Geometric edge meshes

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We consider the Stokes problem of incompressible fluid flow in three-dimensional polyhedral domains discretized on hexahedral meshes with  $hp$ -discontinuous Galerkin finite elements of type  $\mathbb{Q}_k$  for the velocity and  $\mathbb{Q}_{k-1}$  for the pressure. We prove that these elements are inf-sup stable on geometric edge meshes that are refined anisotropically and non-quasiuniformly towards edges and corners. The discrete inf-sup constant is shown to be independent of the aspect ratio of the anisotropic elements and is of  $\mathcal{O}(k^{-3/2})$  in the polynomial degree  $k$ , as in the case of conforming  $\mathbb{Q}_k - \mathbb{Q}_{k-2}$  approximations on the same meshes.

*Keywords:* discontinuous Galerkin methods;  $hp$ -FEM; geometric edge meshes.

### 1. Introduction

It is well known that solutions of elliptic boundary value problems in polyhedral domains exhibit corner and edge singularities. In addition, boundary layers may also arise in laminar, viscous, incompressible flows with moderate Reynolds numbers at faces, edges, and corners. Suitably graded meshes, geometrically refined towards corners, edges, and/or faces, are required in order to achieve an exponential rate of convergence of  $hp$ -finite element approximations; see, e.g. Andersson *et al.* (1995), Babuška & Guo (1996), Melenk & Schwab (1998), Schwab & Suri (1996), Schwab *et al.* (1998), and the references cited therein.

The stationary Stokes and Navier–Stokes equations are mixed elliptic systems with saddle point variational structure. The stability and accuracy of the corresponding finite-element approximations depend on an inf-sup condition for the finite-element spaces that are chosen for the velocity and the pressure. Even for stable velocity–pressure combinations, the corresponding inf-sup constants may in general be very sensitive to the

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aspect ratio of the mesh, thus degrading stability if very thin elements are employed, as required for the resolution of boundary layers and edge singularities. It has recently been shown for two- and three-dimensional conforming approximations employing  $\mathbb{Q}_k - \mathbb{Q}_{k-2}$  elements, on corner, edge, and boundary-layer tensor product meshes of hexahedra, that the dependence on the polynomial degree of the inf-sup constant for the Stokes problem might be only slightly worse than that for isotropically refined triangulations but is independent of the aspect ratio of the anisotropic elements; see Schötzau & Schwab (1998), Schötzau *et al.* (1999), Ainsworth & Coggins (2000), and Toselli & Schwab (2003).

Discontinuous Galerkin (DG) approximations rely on discrete spaces consisting of piecewise polynomial functions with no continuity constraints across the interfaces between the elements of a triangulation. They present considerable advantages for certain types of problems, especially those modelling phenomena where convection is strong; see e.g. Cockburn (1999), Cockburn *et al.* (2000), Cockburn & Shu (2001), and references therein. DG approximations often allow for greater flexibility in the design of the mesh and in the choice of the approximation spaces since they do not usually require geometrically conforming triangulations. We note, however, that even if convection is the dominant effect of a problem, diffusive terms still need to be accounted for and correctly discretized in a DG framework. Several mixed DG approximations have been proposed for incompressible fluid flow. We mention the approaches of Baker *et al.* (1990), Karakashian & Jureidini (1998), Cockburn *et al.* (2002), Cockburn *et al.* (2003), Hansbo & Larson (2002), and Girault *et al.* (2002). In Toselli (2002) and Schötzau *et al.* (2003), DF *hp*-approximations in two and three dimensions have been proposed and analysed for tensor product meshes. Numerical evidence hints that DG approximations exhibit better divergence stability properties than the corresponding conforming ones; see Toselli (2002) for the case of discontinuous  $\mathbb{Q}_k - \mathbb{Q}_k$ ,  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$ , and  $\mathbb{Q}_k - \mathbb{Q}_{k-2}$  elements.

In this paper, we consider  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$  DG approximations in three dimensions. They were originally studied by Toselli (2002) and then by Schötzau *et al.* (2003) for shape-regular meshes, possibly with hanging nodes. In particular, it was shown that these approximation spaces are divergence stable uniformly with respect to the mesh size  $h$ . The best bound for the inf-sup constant in terms of the polynomial degree  $k$  was given by Schötzau *et al.* (2003) and decreases as  $k^{-1}$  both in two and three dimensions. Even though this estimate does not appear to be sharp, at least in two dimensions (see the numerical results in Toselli, 2002), it ensures the same  $p$ -version convergence rate for the velocity and the pressure as that of conforming  $\mathbb{Q}_k - \mathbb{Q}_{k-2}$  elements in three dimensions, but with a gap in the polynomial degree of the velocity–pressure pair of just one. We also note that a similar approach was considered in Hansbo & Larson (2002) for  $h$ -version finite element approximations on shape-regular tetrahedral meshes for mixed formulations of elasticity problems.

Here, we generalize our analysis in Schötzau *et al.* (2003) to the case of geometric edge meshes consisting of hexahedral elements in  $\mathbb{R}^3$ . These meshes are refined anisotropically and non-quasiuniformly towards edges and corners in order to resolve edge and corner singularities at exponential rates of convergence. We show that the inf-sup constant for discontinuous  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$  elements decreases as  $Ck^{-3/2}$ , with a constant  $C$  that only depends on the geometric grading factor, and is independent of the degree  $k$ , the level of refinement, and the aspect ratio of the anisotropic elements. We recall that for conforming  $\mathbb{Q}_k - \mathbb{Q}_{k-2}$  approximations the inf-sup constant on geometric edge meshes has been shown

to decrease as  $Ck^{-1/2}$  in two dimensions and as  $Ck^{-3/2}$  in three dimensions; see Schötzau & Schwab (1998), Schötzau *et al.* (1999), Toselli & Schwab (2003) and the references therein. The inf-sup constant of our method has thus the same dependence on  $k$  as that of conforming approximations, but with an optimal gap of just one degree between the velocity and the pressure approximation.

For simplicity, we assume throughout that the geometric meshes consist of stretched affine hexahedra. While hexahedral elements are essential in our stability proofs, the condition that the element maps be affine may be weakened to the extent that the meshes are patchwise mapped from suitable reference patches by smooth, bijective and nondegenerate maps. In this case, the velocity spaces need to be suitably adapted in the physical coordinates as in Chilton & Suri (2000), but our stability results on anisotropic meshes still apply in the reference patches.

We consider here the symmetric interior penalty DG method, but emphasize that our stability results remain valid for all the methods discussed in Schötzau *et al.* (2003). Note that our analysis is also valid for  $hp$ -DGFEM approximations of elasticity problems in nearly incompressible materials, see, e.g. Brezzi & Fortin (1991), and Franca & Stenberg (1991), since the same inf-sup condition is required in order to have approximations that remain stable close to the incompressible limit.

This paper is organized as follows: in Section 2, we review the discrete setting from Schötzau *et al.* (2003) that we use in our stability analysis. Section 3 is devoted to the definition and construction of geometric edge meshes. In Section 4, we establish continuity and coercivity properties of the DG forms. Our main stability result is an inf-sup condition for the  $hp$ -discretization of the divergence form on geometric edge meshes; it is presented in Section 5. In order to prove this result, several ingredients are needed. First, in Section 6, we establish a macro-element technique for mixed  $hp$ -discontinuous Galerkin discretizations in the spirit of Stenberg & Suri (1996), Schötzau & Schwab (1998), Schötzau *et al.* (1999), and Toselli & Schwab (2003). This technique allows us to reduce the investigation of divergence stability to certain reference configurations which we refer to as patches. Then, to address the stability on one of these configurations, namely the edge patch, we provide estimates of Raviart–Thomas interpolants on stretched hexahedra in Section 7. The stability on edge patches is shown in Section 8. Finally, we complete the proof of our stability result in Section 9.

## 2. Mixed $hp$ -DGFEM for the Stokes problem

In this section, we introduce mixed  $hp$ -discontinuous Galerkin methods for the Stokes problem of incompressible fluid flow, and review the theoretical framework of Schötzau *et al.* (2003) that we use to analyse the methods on geometric edge meshes.

### 2.1 The Stokes equations

Let  $\Omega$  be a bounded polyhedral domain in  $\mathbb{R}^3$ , with  $\mathbf{n}$  denoting the outward normal unit vector to its boundary  $\partial\Omega$ . Given a source term  $\mathbf{f} \in L^2(\Omega)^3$  and a Dirichlet datum  $\mathbf{g} \in H^{1/2}(\partial\Omega)^3$  satisfying the compatibility condition  $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, ds = 0$ , the Stokes problem of

incompressible fluid flow consists in finding a velocity field  $\mathbf{u}$  and a pressure  $p$  such that

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

By setting  $\mathbf{V} := H^1(\Omega)^3$ ,  $Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q \, d\mathbf{x} = 0\}$  and

$$A(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nu \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x}, \quad B(\mathbf{v}, q) = - \int_{\Omega} q \, \nabla \cdot \mathbf{v} \, d\mathbf{x},$$

we obtain the usual mixed variational formulation of the Stokes problem that consists in finding  $(\mathbf{u}, p) \in \mathbf{V} \times Q$ , with  $\mathbf{u} = \mathbf{g}$  on  $\partial\Omega$ , such that

$$\begin{cases} A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} \\ B(\mathbf{u}, q) = 0 \end{cases} \quad (2.2)$$

for all  $\mathbf{v} \in H_0^1(\Omega)^3$  and  $q \in Q$ . As usual,  $H_0^1(\Omega)^3$  is the subspace of  $H^1(\Omega)^3$  of vectors that vanish on  $\partial\Omega$ .

The well-posedness of (2.2) is ensured by the continuity of  $A(\cdot, \cdot)$  and  $B(\cdot, \cdot)$ , the coercivity of  $A(\cdot, \cdot)$ , and the following inf-sup condition:

$$\inf_{0 \neq q \in L_0^2(\Omega)} \sup_{0 \neq \mathbf{v} \in H_0^1(\Omega)^d} \frac{B(\mathbf{v}, q)}{|\mathbf{v}|_1 \|q\|_0} \geq \gamma_{\Omega} > 0, \quad (2.3)$$

with an inf-sup constant  $\gamma_{\Omega}$  only depending on  $\Omega$ ; see e.g. Brezzi & Fortin (1991) and Girault & Raviart (1986). Here, we denote by  $\|\cdot\|_{s, \mathcal{D}}$  and  $|\cdot|_{s, \mathcal{D}}$  the norm and seminorm of  $H^s(\mathcal{D})$  and  $H^s(\mathcal{D})^3$ ,  $s \geq 0$ . When  $\mathcal{D} = \Omega$ , we drop the subscript.

## 2.2 Meshes and trace operators

Throughout, we consider meshes  $\mathcal{T}$  in two and three space dimensions that consist of quadrilaterals and hexahedra  $\{K\}$ , respectively. Each element  $K \in \mathcal{T}$  is affinely equivalent to a reference element  $\widehat{K}$ , which is either the reference square  $\widehat{S} = (-1, 1)^2$  or the reference cube  $\widehat{Q} = (-1, 1)^3$ . The edges of  $\widehat{S}$  and the faces of  $\widehat{Q}$  are denoted by  $\widehat{f}_m$ ,  $m = 1, \dots, 2d$ ,  $d = 2, 3$ , where

$$\begin{aligned} \widehat{f}_1 &= \{x = -1\}, & \widehat{f}_2 &= \{x = 1\}, \\ \widehat{f}_3 &= \{y = -1\}, & \widehat{f}_4 &= \{y = 1\}, \\ \widehat{f}_5 &= \{z = -1\}, & \widehat{f}_6 &= \{z = 1\}, \end{aligned} \quad d = 3.$$

We write  $\{f_i\}_{i=1}^{2d}$  to denote the edges or faces of an element  $K \in \mathcal{T}$ ; they are obtained by mapping the corresponding ones of  $\widehat{K}$ . In general, we allow for *irregular* meshes, i.e. meshes with so-called hanging nodes (see Schwab, 1998, Section 4.4.1), but suppose that the intersection between neighbouring elements is a vertex, an edge, or a face (if  $d = 3$ ) of at least one of the two elements. For an element  $K \in \mathcal{T}$ , we denote by  $h_K$  its diameter and

by  $\rho_K$  the radius of the largest circle or sphere that can be inscribed into  $K$ . A mesh  $\mathcal{T}$  is called *shape-regular* if

$$h_K \leq c \rho_K \quad \forall K \in \mathcal{T}, \quad (2.4)$$

for a shape-regularity constant  $c > 0$  that is independent of the elements. Our meshes are not necessarily shape-regular; see Section 3.

Let now  $\mathcal{T}$  be a hexahedral mesh on  $\Omega$ . An interior face of  $\mathcal{T}$  is the (non-empty) two-dimensional interior of  $\partial K^+ \cap \partial K^-$ , where  $K^+$  and  $K^-$  are two adjacent elements of  $\mathcal{T}$ . Similarly, a boundary face of  $\mathcal{T}$  is the (non-empty) two-dimensional interior of  $\partial K \cap \partial \Omega$  which consists of entire faces of  $\partial K$ . We denote by  $\mathcal{E}_{\mathcal{T}}$  the union of all interior faces of  $\mathcal{T}$ , by  $\mathcal{E}_{\mathcal{B}}$  the union of all boundary faces, and set  $\mathcal{E} = \mathcal{E}_{\mathcal{T}} \cup \mathcal{E}_{\mathcal{B}}$ .

On  $\mathcal{E}$ , we define the following trace operators. First, let  $f \subset \mathcal{E}_{\mathcal{T}}$  be an interior face shared by two elements  $K^+$  and  $K^-$ . Let  $\mathbf{v}$ ,  $q$ , and  $\underline{\tau}$  be vector-, scalar- and matrix-valued functions, respectively, that are smooth inside each element  $K^\pm$ , and let us denote by  $\mathbf{v}^\pm$ ,  $q^\pm$  and  $\underline{\tau}^\pm$  the traces of  $\mathbf{v}$ ,  $q$  and  $\underline{\tau}$  on  $f$  from the interior of  $K^\pm$ . We define the mean values and the normal jumps at  $\mathbf{x} \in f$  as

$$\begin{aligned} \{\{\mathbf{v}\}\} &:= (\mathbf{v}^+ + \mathbf{v}^-)/2, & \llbracket \mathbf{v} \rrbracket &:= \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-}, \\ \{\{q\}\} &:= (q^+ + q^-)/2, & \llbracket q \rrbracket &:= q^+ \mathbf{n}_{K^+} + q^- \mathbf{n}_{K^-}, \\ \{\{\underline{\tau}\}\} &:= (\underline{\tau}^+ + \underline{\tau}^-)/2, & \llbracket \underline{\tau} \rrbracket &:= \underline{\tau}^+ \mathbf{n}_{K^+} + \underline{\tau}^- \mathbf{n}_{K^-}. \end{aligned}$$

Here, we denote by  $\mathbf{n}_K$  the outward normal unit vector to the boundary  $\partial K$  of an element  $K$ . We also need to define the matrix-valued jump of  $\mathbf{v}$ , namely

$$\llbracket \mathbf{v} \rrbracket := \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-},$$

where, for two vectors  $\mathbf{a}$  and  $\mathbf{b}$ ,  $[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i b_j$ . On a boundary face  $f \subset \mathcal{E}_{\mathcal{B}}$  given by  $f = \partial K \cap \partial \Omega$ , we then set accordingly  $\{\{\mathbf{v}\}\} := \mathbf{v}$ ,  $\{\{q\}\} := q$ ,  $\{\{\underline{\tau}\}\} := \underline{\tau}$ , as well as  $\llbracket \mathbf{v} \rrbracket := \mathbf{v} \cdot \mathbf{n}$ ,  $\llbracket \mathbf{v} \rrbracket := \mathbf{v} \otimes \mathbf{n}$ ,  $\llbracket q \rrbracket := q \mathbf{n}$  and  $\llbracket \underline{\tau} \rrbracket := \underline{\tau} \mathbf{n}$ .

### 2.3 Finite-element spaces

For a mesh  $\mathcal{T}$  on a polyhedron  $\mathcal{D}$  and an approximation order  $k \geq 0$ , we introduce the finite-element spaces

$$\begin{aligned} \mathbf{V}_h^k(\mathcal{T}; \mathcal{D}) &:= \{ \mathbf{v} \in L^2(\mathcal{D})^3 : \mathbf{v}|_K \in \mathbb{Q}_k(K)^3, K \in \mathcal{T} \}, \\ \mathcal{Q}_h^k(\mathcal{T}; \mathcal{D}) &:= \{ q \in L^2(\mathcal{D}) : q|_K \in \mathbb{Q}_k(K), K \in \mathcal{T}, \int_{\mathcal{D}} q \, d\mathbf{x} = 0 \}, \end{aligned}$$

where  $\mathbb{Q}_k(K)$  is the space of polynomials of maximum degree  $k$  in each variable on the element  $K$ . Further, we define the subspace  $\tilde{\mathbf{V}}_h^k(\mathcal{T}; \mathcal{D})$  of  $\mathbf{V}_h^k(\mathcal{T}; \mathcal{D})$  of vectors with vanishing normal component on the boundary of  $\mathcal{D}$

$$\tilde{\mathbf{V}}_h^k(\mathcal{T}; \mathcal{D}) = \{ \mathbf{v} \in \mathbf{V}_h^k(\mathcal{T}; \mathcal{D}) : \mathbf{v} \cdot \mathbf{n}_{\mathcal{D}} = 0 \text{ on } \partial \mathcal{D} \},$$

with  $\mathbf{n}_{\mathcal{D}}$  denoting the outward normal unit vector to  $\partial \mathcal{D}$ . For  $\mathcal{D} = \Omega$ , we omit the dependence on the domain and simply write  $\mathbf{V}_h^k(\mathcal{T})$ ,  $\mathcal{Q}_h^k(\mathcal{T})$  and  $\tilde{\mathbf{V}}_h^k(\mathcal{T})$ .

#### 2.4 Mixed discontinuous Galerkin approximations

For a mesh  $\mathcal{T}$  on  $\Omega$ , we approximate the velocity and pressure in the spaces  $\mathbf{V}_h$  and  $Q_h$  given by

$$\mathbf{V}_h := \mathbf{V}_h^k(\mathcal{T}), \quad Q_h := Q_h^{k-1}(\mathcal{T}), \quad k \geq 1.$$

We refer to this velocity–pressure pair as (discontinuous)  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$  elements.

In order to apply the framework of Schötzau *et al.* (2003), we need to define the additional space  $\mathbf{V}(h) := \mathbf{V} + \mathbf{V}_h$ , endowed with the broken norm

$$\|\mathbf{v}\|_h^2 := \sum_{K \in \mathcal{T}} |\mathbf{v}|_{1,K}^2 + \int_{\mathcal{E}} \delta \|\llbracket \mathbf{v} \rrbracket\|^2 ds, \quad \mathbf{v} \in \mathbf{V}(h).$$

Here,  $\delta \in L^\infty(\mathcal{E})$  is the so-called discontinuity stabilization function, for which we will make a precise choice in Section 3.2.

Next, we introduce the auxiliary space

$$\underline{\Sigma}_h := \{ \underline{\tau} \in L^2(\Omega)^{3 \times 3} : \underline{\tau}|_K \in \mathbb{Q}_k(K)^{3 \times 3}, K \in \mathcal{T} \},$$

and define the lifting operators  $\underline{\mathcal{L}} : \mathbf{V}(h) \rightarrow \underline{\Sigma}_h$  and  $\mathcal{M} : \mathbf{V}(h) \rightarrow Q_h$  by

$$\int_{\Omega} \underline{\mathcal{L}}(\mathbf{v}) : \underline{\tau} \, d\mathbf{x} = \int_{\mathcal{E}} \llbracket \mathbf{v} \rrbracket : \{\{\underline{\tau}\}\} \, ds \quad \forall \underline{\tau} \in \underline{\Sigma}_h, \quad (2.5)$$

$$\int_{\Omega} \mathcal{M}(\mathbf{v}) q \, d\mathbf{x} = \int_{\mathcal{E}} \llbracket \mathbf{v} \rrbracket \{\{q\}\} \, ds \quad \forall q \in Q_h. \quad (2.6)$$

We consider the following mixed DG method: find  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$  such that

$$\begin{cases} A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) = F_h(\mathbf{v}) \\ B_h(\mathbf{u}_h, q) = G_h(q) \end{cases} \quad (2.7)$$

for all  $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$ . Here,  $A_h : \mathbf{V}(h) \times \mathbf{V}(h) \rightarrow \mathbb{R}$  and  $B_h : \mathbf{V}(h) \times Q \rightarrow \mathbb{R}$  have the following forms:

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \nu [\nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{u}) : \nabla_h \mathbf{v} - \underline{\mathcal{L}}(\mathbf{v}) : \nabla_h \mathbf{u}] \, d\mathbf{x} \\ &\quad + \nu \int_{\mathcal{E}} \delta \llbracket \mathbf{u} \rrbracket : \llbracket \mathbf{v} \rrbracket \, ds, \\ B_h(\mathbf{v}, q) &= - \int_{\Omega} q [\nabla_h \cdot \mathbf{v} - \mathcal{M}(\mathbf{v})] \, d\mathbf{x}, \end{aligned} \quad (2.8)$$

where  $\nabla_h$  is the discrete gradient, taken elementwise. The functionals  $F_h : \mathbf{V}_h \rightarrow \mathbb{R}$  and  $G_h : Q_h \rightarrow \mathbb{R}$  are given by

$$\begin{aligned} F_h(\mathbf{v}) &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} - \int_{\mathcal{E}_B} (\mathbf{g} \otimes \mathbf{n}) : \{\{ \nu \nabla_h \mathbf{v} \}\} \, ds + \nu \int_{\mathcal{E}_B} \delta \mathbf{g} \cdot \mathbf{v} \, ds, \\ G_h(q) &= \int_{\mathcal{E}_B} q \mathbf{g} \cdot \mathbf{n} \, ds. \end{aligned}$$

Restricted to discrete functions in  $\mathbf{V}_h$  and  $Q_h$ , we have

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \nu \nabla_h \mathbf{u} : \nabla_h \mathbf{v} \, d\mathbf{x} - \int_{\mathcal{E}} (\{\nu \nabla_h \mathbf{v}\} : \llbracket \mathbf{u} \rrbracket + \{\nu \nabla_h \mathbf{u}\} : \llbracket \mathbf{v} \rrbracket) \, ds \\ &\quad + \nu \int_{\mathcal{E}} \delta \llbracket \mathbf{u} \rrbracket : \llbracket \mathbf{v} \rrbracket \, ds, \\ B_h(\mathbf{v}, q) &= - \int_{\Omega} q \nabla_h \cdot \mathbf{v} \, d\mathbf{x} + \int_{\mathcal{E}} \{q\} \llbracket \mathbf{v} \rrbracket \, ds. \end{aligned}$$

We also note that for  $q \in Q_h$  and  $\mathbf{v} \in \mathbf{V}_h \cap H_0(\text{div}; \Omega)$

$$B_h(\mathbf{v}, q) = B(\mathbf{v}, q) = - \int_{\Omega} q \nabla \cdot \mathbf{v} \, d\mathbf{x}, \quad (2.9)$$

where the space  $H_0(\text{div}; \Omega)$  consists of square-integrable vectors with square-integrable divergence and vanishing normal component on  $\partial\Omega$ . Thus, the space  $\mathbf{V}_h \cap H_0(\text{div}; \Omega)$  consists of discrete vectors with continuous normal component across the inter-element boundaries and vanishing normal component on  $\partial\Omega$ ; see e.g. Brezzi & Fortin (1991, Chapter III.3).

REMARK 1 The form  $B_h$  and the functional  $G_h$  are exactly those considered in the mixed DG approaches of Cockburn *et al.* (2002), Hansbo & Larson (2002), Toselli (2002), and Schötzau *et al.* (2003). The form  $A_h$  in (2.8) is the so-called interior penalty (IP) form. Several other choices are possible for  $A_h$ , as discussed in Schötzau *et al.* (2003). All the results of this paper hold verbatim for these other forms as well.

## 2.5 Well-posedness and error estimates

Problem (2.7) was analysed in Schötzau *et al.* (2003) where an abstract framework was introduced.

We assume that the forms  $A_h$  and  $B_h$  satisfy the following continuity properties:

$$A_h(\mathbf{u}, \mathbf{v}) \leq \alpha_1 \|\mathbf{u}\|_h \|\mathbf{v}\|_h, \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}(h), \quad (2.10)$$

$$B_h(\mathbf{v}, q) \leq \alpha_2 \|\mathbf{v}\|_h \|q\|_0, \quad (\mathbf{v}, q) \in \mathbf{V}(h) \times Q, \quad (2.11)$$

with constants  $\alpha_1 > 0$  and  $\alpha_2 > 0$ , and that  $A_h$  is coercive

$$A_h(\mathbf{u}, \mathbf{u}) \geq \beta \|\mathbf{u}\|_h^2, \quad \mathbf{u} \in \mathbf{V}_h, \quad (2.12)$$

for a constant  $\beta > 0$ . Next, we suppose that the following discrete inf-sup condition for the finite-element spaces  $\mathbf{V}_h$  and  $Q_h$  holds:

$$\inf_{0 \neq q \in Q_h} \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_0} \geq \gamma_h > 0. \quad (2.13)$$

Condition (2.13) is also referred to as divergence stability. Finally, we assume the functionals  $F_h : \mathbf{V}_h \rightarrow \mathbb{R}$  and  $G_h : Q_h \rightarrow \mathbb{R}$  to be continuous.

The above conditions ensure the well-posedness of (2.7). Indeed, (2.7) has a unique solution and we have the following error bounds from Sections 3 and 4 of Schötzau *et al.* (2003), with  $(\mathbf{u}, p)$  denoting the exact solution of (2.1):

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_h &\leq C[\gamma_h^{-1} \inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_h + \inf_{q \in Q_h} \|p - q\|_0 + \mathcal{R}_h(\mathbf{u}, p)], \\ \|p - p_h\|_0 &\leq C[\gamma_h^{-1} \inf_{q \in Q_h} \|p - q\|_0 + \gamma_h^{-2} \inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_h + \gamma_h^{-1} \mathcal{R}_h(\mathbf{u}, p)],\end{aligned}\quad (2.14)$$

where the constants  $C$  only depend on  $\alpha_1$ ,  $\alpha_2$  and  $\beta$ , and where  $\mathcal{R}_h(\mathbf{u}, p)$  is the residual defined by

$$\mathcal{R}_h(\mathbf{u}, p) := \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h} \frac{|A_h(\mathbf{u}, \mathbf{v}) + B_h(\mathbf{v}, p) - F_h(\mathbf{v})|}{\|\mathbf{v}\|_h}. \quad (2.15)$$

(Note that  $B_h(\mathbf{u}, q) = G_h(q)$  for all  $q \in Q_h$ .)

In Schötzau *et al.* (2003), the above conditions have been verified on isotropically refined, shape-regular meshes in two and three dimensions. It has then been proven in Theorem 9.1 there that, for  $\delta$  of the order  $k^2/h$  and piecewise smooth solutions, the estimates in (2.14) lead to algebraic convergence rates that are optimal in the mesh sizes and slightly suboptimal in the polynomial degrees. In particular, the residual  $\mathcal{R}_h$  in (2.15) has been shown to be optimally convergent in the mesh sizes and the polynomial degrees; see Schötzau *et al.* (2003, Proposition 8.1). Moreover, the recent work of Schötzau & Wihler (2002) has shown that, for Stokes flow in polygonal domains, the error estimates (2.14) give rise to exponential rates of convergence on geometrically refined shape-regular meshes.

In the following, we generalize the stability results of Schötzau *et al.* (2003) to three-dimensional geometric edge meshes, which are highly anisotropic. In particular, we show that the forms in (2.8) satisfy the above conditions on such meshes with constants  $\alpha_1$ ,  $\alpha_2$ ,  $\beta$  and  $\gamma_h$  that can be bounded independently of the aspect ratio of the anisotropic elements, provided that  $\delta$  is suitably chosen. Geometric edge meshes are introduced in Section 3. Continuity and coercivity properties are then shown in Section 4. The crucial stability result is the discrete inf-sup condition in Section 5.

### 3. Geometric edge meshes

In this section, we introduce a class of *geometric meshes* designed to resolve corner and edge singularities that arise in Stokes flow or nearly incompressible elasticity. These meshes are referred to as *geometric edge meshes*; they are, roughly speaking, tensor products of meshes that are geometrically refined towards the edges.

#### 3.1 Construction of geometric edge meshes

Geometric edge meshes are determined by a *mesh grading factor*  $\sigma \in (0, 1)$  and a *number of layers*  $n$ , the thinnest layer having width proportional to  $\sigma^n$ . We recall that exponential convergence of *hp*-finite element approximations is achieved if  $n$  is suitably chosen. For singularity resolution,  $n$  is required to be proportional to the polynomial degree  $k$ ; see Andersson *et al.* (1995) and Babuška & Guo (1996).



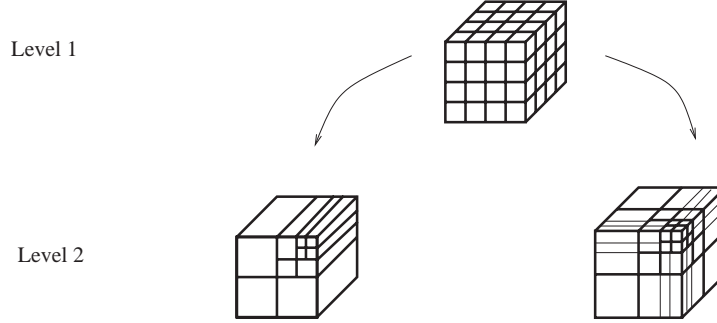


FIG. 1. Hierarchical structure of a geometric edge mesh  $\mathcal{T}^{n,\sigma}$ . The macro-elements  $M$  at the boundary of  $\Omega$  (level 1) are further refined as edge and corner patches (level 2). Here we have chosen  $\sigma = 0.5$  and  $n = 3$ .

On  $\Omega$ , a *geometric edge mesh*  $\mathcal{T}^{n,\sigma}$  is constructed by considering an initial shape-regular macro-triangulation  $\mathcal{T}_m = \{M\}$  of  $\Omega$ , possibly consisting of just one element. The macro-elements  $M$  in the interior of  $\Omega$  can be refined isotropically and regularly (not discussed further) while the macro-elements  $M$  at the boundary of  $\Omega$  are refined geometrically and anisotropically towards edges and corners. This geometric refinement is obtained by affinely mapping reference triangulations (referred to as *patches*) on  $\widehat{Q}$  onto the macro-elements  $M$  using elemental maps  $F_M : \widehat{Q} \rightarrow M$ . An illustration of this process is shown in Fig. 1. For edge meshes, the following patches on  $\widehat{Q} = \widehat{I}^3$ ,  $\widehat{I} = (-1, 1)$ , are used for the geometric refinement towards the boundary of  $\Omega$ :

- Edge patches: an edge patch  $\mathcal{T}_e^{n,\sigma}$  on  $\widehat{Q}$  is given by

$$\mathcal{T}_e^{n,\sigma} := \{K_{xy} \times \widehat{I} \mid K_{xy} \in \mathcal{T}_{xy}^{n,\sigma}\},$$

where  $\mathcal{T}_{xy}^{n,\sigma}$  is an irregular corner mesh, geometrically refined towards a vertex of  $\widehat{S} = \widehat{I}^2$  with grading factor  $\sigma$  and  $n$  layers of refinement; see Fig. 1 (level 2, left).

- Corner patches: in order to build a corner patch  $\mathcal{T}_c^{n,\sigma}$  on  $\widehat{Q}$ , we first consider an initial, irregular, corner mesh  $\mathcal{T}_{c,m}^{n,\sigma}$ , geometrically refined towards a vertex of  $\widehat{Q}$ , with grading factor  $\sigma$  and  $n$  layers of refinement; see the mesh in bold lines in Fig. 1 (level 2, right). The elements of this mesh are then irregularly refined towards the three edges adjacent to the vertex in order to obtain the mesh  $\mathcal{T}_c^{n,\sigma}$ .

For simplicity, we always assume that the only hanging nodes contained in geometric edge meshes  $\mathcal{T}^{n,\sigma}$  are those contained in the edge and corner patches.

The geometric edge meshes satisfy the following property; see also Gerdes *et al.* (2001).

**PROPERTY 2** Let  $\mathcal{T}^{n,\sigma}$  be a geometric edge mesh on  $\Omega$  and  $K \in \mathcal{T}^{n,\sigma}$ . Then  $K$  can be written as  $K = F_K(K_{xyz})$ , where  $K_{xyz}$  is of the form

$$K_{xyz} = I_x \times I_y \times I_z = (x_1, x_2) \times (y_1, y_2) \times (z_1, z_2),$$

and  $F_K$  is an affine mapping, the Jacobian of which satisfies

$$|\det(J_K)| \leq C, \quad |\det(J_K^{-1})| \leq C,$$

with  $C$  only depending on the angles of  $K$  but not on its dimensions.

We note that the constants in Property 2 only depend on the constant in (2.4) for the underlying macro-element mesh  $\mathcal{T}_m$ . The dimensions of  $K_{xyz}$  on the other hand may depend on the geometric grading factor and the number of refinements.

For an element  $K$  of a geometric edge mesh, we define, according to Property 2,

$$h_x^K = h_x = x_2 - x_1, \quad h_y^K = h_y = y_2 - y_1, \quad h_z^K = h_z = z_2 - z_1.$$

### 3.2 Discontinuity stabilization on geometric meshes

In this section, we define the discontinuity stabilization parameter  $\delta \in L^\infty(\mathcal{E})$  on geometric edge meshes. We note that this approach was originally proposed in Georgoulis & Süli (2001). Let  $f$  be an entire face of an element  $K$  of a geometric edge mesh  $\mathcal{T}^{n,\sigma}$  on  $\Omega$ . According to Property 2,  $K$  can be obtained from a stretched parallelepiped  $K_{xyz}$  by an affine mapping  $F_K$  that only changes the angles. Suppose that the face  $f$  is the image of, for example, the face  $\{x = x_1\}$ . We set  $h_f = h_x$ . For a face perpendicular to the  $y$ - or  $z$ -direction, we choose  $h_f = h_y$  or  $h_f = h_z$ .

Let now  $K$  and  $K'$  be two elements with entire faces  $f$  and  $f'$  that share an interior face, e.g.  $f = f \cap f'$  in  $\mathcal{E}_{\mathcal{I}}$ . We have

$$ch_f \leq h_{f'} \leq c^{-1}h_f, \quad (3.1)$$

with a constant  $c > 0$  that only depends on the geometric grading factor  $\sigma$  and the constant in (2.4) for the underlying macro-element mesh  $\mathcal{T}_m$ . We then define the function  $h \in L^\infty(\mathcal{E})$  by

$$h(\mathbf{x}) := \begin{cases} \min\{h_f, h_{f'}\} & \mathbf{x} \in f \cap f' \subset \mathcal{E}_{\mathcal{I}}, \\ h_f & \mathbf{x} \in f \subset \mathcal{E}_{\mathcal{B}}. \end{cases} \quad (3.2)$$

Furthermore, we define

$$\delta(\mathbf{x}) = \delta_0 h^{-1}(\mathbf{x}) k^2, \quad (3.3)$$

with a parameter  $\delta_0 > 0$  that is independent of  $h$  and  $k$ .

REMARK 3 For isotropically refined, shape-regular meshes, the definition in (3.3) is equivalent to the usual definition of  $\delta$ , see Schötzau *et al.* (2003).

Strongly related to the choice of  $\delta$  in (3.2) is the following discrete trace inequality.

LEMMA 4 Let  $K$  be an element of a geometric edge mesh  $\mathcal{T}^{n,\sigma}$  on  $\Omega$  and  $f$  an entire face of  $K$ . Then

$$\|\varphi\|_{0,f}^2 \leq Ch_f^{-1} \max\{1, k\}^2 \|\varphi\|_{0,K}^2$$

for any  $\varphi \in \mathbb{Q}_k(K)$ ,  $k \geq 0$ , with a constant only depending on the constants in Property 2.

*Proof.* First we note that on the reference cube  $\widehat{Q}$ , this estimate follows from standard inverse inequalities, see e.g. Schwab (1998, Theorem 4.76). Next, let  $K = K_{xyz} =$

$(x_1, x_2) \times (y_1, y_2) \times (z_1, z_2)$  be an axiparallel element. We may assume that the face  $f$  is given by  $f_{yz} = \{x_1\} \times (y_1, y_2) \times (z_1, z_2)$ . A simple scaling argument then yields

$$\|\varphi\|_{0,f_{yz}}^2 \leq Ch_x^{-1} \max\{1, k\}^2 \|\varphi\|_{0,K_{xyz}}^2 \quad (3.4)$$

for any  $\varphi \in \mathbb{Q}_k(K_{xyz})$ , with  $h_x = x_2 - x_1$  and a constant  $C > 0$ . Finally, since an element  $K$  of a geometric edge mesh can be written as  $K = F_K(K_{xyz})$  according to Property 2, the claim follows from (3.4) by a scaling argument that takes into account the definition of  $h_f$ .  $\square$

#### 4. Continuity and coercivity on geometric edge meshes

We first establish the continuity of  $A_h$  and  $B_h$  as well as the coercivity of  $A_h$  on geometric edge meshes.

**THEOREM 5** Let  $\mathcal{T}^{n,\sigma}$  be a geometric edge mesh on  $\Omega$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Let the discontinuity stabilization function  $\delta$  be defined as in (3.2) and (3.3).

The forms  $A_h$  and  $B_h$  in (2.8) are continuous,

$$\begin{aligned} |A_h(\mathbf{v}, \mathbf{w})| &\leq \nu \alpha_1 \|\mathbf{v}\|_h \|\mathbf{w}\|_h & \forall \mathbf{v}, \mathbf{w} \in \mathbf{V}(h), \\ |B_h(\mathbf{v}, q)| &\leq \alpha_2 \|\mathbf{v}\|_h \|q\|_0 & \forall \mathbf{v} \in \mathbf{V}(h), q \in Q, \end{aligned}$$

with continuity constants  $\alpha_1 > 0$  and  $\alpha_2 > 0$  that depend on  $\delta_0$  and the constants in Property 2, but are independent of  $\nu, k, n$ , and the aspect ratio of the anisotropic elements in  $\mathcal{T}^{n,\sigma}$ .

Furthermore, there exists a constant  $\delta_{\min} > 0$  that depends on the constants in Property 2, but is independent of  $\nu, k, n$ , and the aspect ratio of the anisotropic elements in  $\mathcal{T}^{n,\sigma}$ , such that, for any  $\delta_0 \geq \delta_{\min}$ ,

$$A_h(\mathbf{v}, \mathbf{v}) \geq \nu \beta \|\mathbf{v}\|_h^2 \quad \forall \mathbf{v} \in \mathbf{V}_h,$$

for a coercivity constant  $\beta > 0$  depending on  $\delta_0$  and the constants in Property 2, but independent of  $\nu, k, n$ , and the aspect ratio of the anisotropic elements in  $\mathcal{T}^{n,\sigma}$ .

*Proof.* We first claim that the lifting operators  $\underline{\mathcal{L}}$  and  $\mathcal{M}$  in (2.5) and (2.6) satisfy

$$\|\underline{\mathcal{L}}(\mathbf{v})\|_0^2 \leq C \int_{\mathcal{E}} \delta \|\llbracket \mathbf{v} \rrbracket\|^2 ds, \quad \|\mathcal{M}(\mathbf{v})\|_0^2 \leq C \int_{\mathcal{E}} \delta \|\llbracket \mathbf{v} \rrbracket\|^2 ds, \quad (4.1)$$

for any  $\mathbf{v} \in \mathbf{V}(h)$ , with  $C > 0$  independent of  $k, n$ , and the aspect ratio of the anisotropic elements.

We show the first estimate in (4.1); the proof of the second one is completely analogous

by noting that  $|\llbracket \mathbf{v} \rrbracket|^2 \leq |\llbracket \mathbf{v} \rrbracket|^2$ . For  $\mathbf{v} \in \mathbf{V}(h)$ , we have

$$\begin{aligned} \|\underline{\mathcal{L}}(\mathbf{v})\|_0 &= \sup_{\underline{\tau} \in \underline{\Sigma}_h} \frac{\int_{\Omega} \underline{\mathcal{L}}(\mathbf{v}) : \underline{\tau} \, d\mathbf{x}}{\|\underline{\tau}\|_0} = \sup_{\underline{\tau} \in \underline{\Sigma}_h} \frac{\int_{\mathcal{E}} \llbracket \mathbf{v} \rrbracket : \{\{\underline{\tau}\}\} \, ds}{\|\underline{\tau}\|_0} \\ &\leq \sup_{\underline{\tau} \in \underline{\Sigma}_h} \frac{(\int_{\mathcal{E}} \delta |\llbracket \mathbf{v} \rrbracket|^2 \, ds)^{\frac{1}{2}} (\int_{\mathcal{E}} \delta^{-1} |\{\{\underline{\tau}\}\}|^2 \, ds)^{\frac{1}{2}}}{\|\underline{\tau}\|_0} \\ &\leq C \sup_{\underline{\tau} \in \underline{\Sigma}_h} \frac{(\int_{\mathcal{E}} \delta |\llbracket \mathbf{v} \rrbracket|^2 \, ds)^{\frac{1}{2}} (\sum_{K \in \mathcal{T}_h} \int_{\partial K} \delta^{-1} |\underline{\tau}|^2 \, ds)^{\frac{1}{2}}}{\|\underline{\tau}\|_0}. \end{aligned}$$

Here, we used the definition of  $\underline{\mathcal{L}}$  and the Cauchy–Schwarz inequality. Since for  $\underline{\tau} \in \underline{\Sigma}_h$

$$\int_{\partial K} \delta^{-1} |\underline{\tau}|^2 \, ds \leq C \sum_{m=1}^6 h_{f_m} k^{-2} \|\underline{\tau}\|_{0,f_m}^2 \leq C \|\underline{\tau}\|_{0,K}^2,$$

thanks to the definition of  $\delta$  and Lemma 4, we obtain the desired estimate for  $\underline{\mathcal{L}}$ .

The continuity of the forms  $A_h$  and  $B_h$  follows immediately from (4.1) and Cauchy–Schwarz inequalities. The coercivity of  $A_h$  can be proven by employing the first estimate in (4.1) and the arithmetic–geometric mean inequality  $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ , for all  $\varepsilon > 0$ , see Arnold *et al.* (2001).  $\square$

**REMARK 6** The results in Theorem 5 are based on the anisotropic stability estimates (4.1) for the lifting operators  $\underline{\mathcal{L}}$  and  $\mathcal{M}$ . These operators are identical for all the DG forms considered in Schötzau *et al.* (2003) and, thus, the results in Theorem 5 hold true for all the forms there as well. We also note that the restriction on  $\delta_0$  is typical for the interior penalty form  $A_h$  and can be avoided if  $A_h$  is chosen to be, for example, the local DG form, the nonsymmetric interior penalty form or the second Bassi–Rebay form, see Schötzau *et al.* (2003).

Next, we address the continuity of  $F_h$  and  $G_h$ .

**THEOREM 7** Let  $\mathcal{T}^{n,\sigma}$  be a geometric edge mesh on  $\Omega$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Let the discontinuity stabilization function  $\delta$  be defined as in (3.2) and (3.3). Then we have

$$\begin{aligned} |F_h(\mathbf{v})| &\leq C [\|\mathbf{f}\|_0 + \nu \|\delta^{\frac{1}{2}} \mathbf{g}\|_{0,\partial\Omega}] \|\mathbf{v}\|_h \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ |G_h(q)| &\leq C \|\delta^{\frac{1}{2}} \mathbf{g}\|_{0,\partial\Omega} \|q\|_0 \quad \forall q \in Q_h, \end{aligned}$$

with continuity constants  $C > 0$  that depend on  $\delta_0$  and the constants in Property 2, but are independent of  $\nu$ ,  $k$ ,  $n$ , and the aspect ratio of the anisotropic elements in  $\mathcal{T}^{n,\sigma}$ .

*Proof.* We first note that we have the Friedrichs inequality

$$\|\mathbf{v}\|_0 \leq C \|\mathbf{v}\|_{1,h} \quad \forall \mathbf{v} \in \mathbf{V}(h), \quad (4.2)$$

with a constant  $C > 0$  depending on  $\delta_0$  and the constants in Property 2. The bound (4.2) follows by proceeding as in the proof in Lemma 3.1 of Lasser & Toselli (2003), taking into

account elliptic regularity theory for polyhedral domains and by using the anisotropic trace inequality

$$\|\varphi\|_{0,f} \leq Ch_f^{-1} \|\varphi\|_{3/2+\varepsilon,K}, \quad \varepsilon > 0,$$

for an element  $K \in \mathcal{T}^{n,\sigma}$  and an entire face  $f$  of  $\partial K$ , with a constant depending on the constants in Property 2.

Let now  $\mathbf{v} \in \mathbf{V}_h$ . From (4.2), we obtain  $|\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}| \leq C \|\mathbf{f}\|_0 \|\mathbf{v}\|_h$ . Further, applying the discrete trace inequality from Lemma 4 as in the proof of Theorem 5,

$$\left| \int_{\mathcal{E}_B} (\mathbf{g} \otimes \mathbf{n}) : \{\nu \nabla_h \mathbf{v}\} \, ds \right| \leq C \nu \|\delta^{\frac{1}{2}} \mathbf{g}\|_{0,\partial\Omega} \|\mathbf{v}\|_h,$$

with a constant depending on  $\delta_0$ , and the constants in Property 2. Finally, the Cauchy–Schwarz inequality yields  $|\nu \int_{\mathcal{E}_B} \delta \mathbf{g} \cdot \mathbf{v} \, ds| \leq \nu \|\delta^{\frac{1}{2}} \mathbf{g}\|_{0,\partial\Omega} \|\mathbf{v}\|_h$ . This proves the assertion for  $F_h$ .

Similarly, for  $q \in Q_h$ ,

$$|G_h(q)| \leq \left| \int_{\mathcal{E}_B} q \mathbf{g} \cdot \mathbf{n} \, ds \right| \leq \|\delta^{\frac{1}{2}} \mathbf{g}\|_{0,\partial\Omega} \left( \int_{\mathcal{E}_B} \delta^{-1} |q|^2 \, ds \right)^{\frac{1}{2}}.$$

Using the trace inequality from Lemma 4 and proceeding as in the proof of Theorem 5, we have  $\int_{\mathcal{E}_B} \delta^{-1} |q|^2 \, ds \leq C \|q\|_0^2$ , with a constant depending on  $\delta_0$ , and the constants in Property 2. This completes the proof.  $\square$

**REMARK 8** The same continuity properties hold for all the functionals  $F_h$  and  $G_h$  in the mixed DG methods analysed in Schötzau *et al.* (2003).

## 5. Divergence stability on geometric edge meshes

Our main result establishes the divergence stability in (2.13) for discontinuous  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$  elements on geometric edge meshes.

**THEOREM 9** Let  $\mathcal{T}^{n,\sigma}$  be a geometric edge mesh on  $\Omega$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Let the discontinuity stabilization function  $\delta$  be defined as in (3.2) and (3.3). Then there exists a constant  $C > 0$  that depends on  $\sigma$ ,  $\delta_0$ , and the shape-regularity of the macro-element mesh, but is independent of  $k$ ,  $n$ , and the aspect ratio of the anisotropic elements in  $\mathcal{T}^{n,\sigma}$ , such that, for any  $n$  and  $k \geq 2$ ,

$$\inf_{0 \neq q \in Q_h^{k-1}(\mathcal{T}^{n,\sigma})} \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h^k(\mathcal{T}^{n,\sigma})} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_0} \geq C k^{-3/2}.$$

Hence, condition (2.13) is satisfied with  $\gamma_h = C k^{-3/2}$ .

**REMARK 10** Theorem 9 shows that the discontinuous  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$  elements considered in this paper are inf-sup stable on geometric edge meshes. It thus extends to the discontinuous Galerkin context the results that were obtained in Schötzau & Schwab (1998), Schötzau

*et al.* (1999), and Toselli & Schwab (2003) for the standard  $\mathbb{Q}_k - \mathbb{Q}_{k-2}$  pair where the velocity space is based on continuous  $\mathbb{Q}_k$  elements and the pressure space on discontinuous  $\mathbb{Q}_{k-2}$  elements. In contrast to this pair, discontinuous  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$  elements are optimally matched with respect to  $h$ -version approximation properties. We further point out that continuous- $\mathbb{Q}_k$ /discontinuous- $\mathbb{Q}_{k-1}$  elements are known to be unstable while continuous- $\mathbb{Q}_k$ /continuous- $\mathbb{Q}_{k-1}$  Hood–Taylor elements are stable; see Brezzi & Falk (1991, Theorems 3.2 and 3.3). However, the dependence of the discrete inf-sup constant on the polynomial degree and the aspect ratio of anisotropic elements seems not to be known for Hood–Taylor elements.

**REMARK 11** The form  $B_h$  is identical for the DG methods of Cockburn *et al.* (2002), Hansbo & Larson (2002), Toselli (2002), and Schötzau *et al.* (2003). Therefore, the stability result in Theorem 9 is valid for all these methods.

The proof of Theorem 9 is carried out in the remaining sections. The first ingredient we need is a macro-element technique that we introduce in Section 6. The second ingredient consists of stability estimates for Raviart–Thomas interpolants on certain anisotropic meshes, derived in Section 7. In Section 8, we establish divergence stability on edge patches. The proof of Theorem 9 is completed in Section 9 by recursively applying the macro-element technique.

## 6. Macro-element technique

In order to prove Theorem 9, we use a macro-element technique; see Stenberg (1990), Stenberg & Suri (1996), Schötzau *et al.* (1999), and Toselli & Schwab (2003). We recall that a geometric edge mesh  $\mathcal{T} = \mathcal{T}^{n,\sigma}$  is obtained by refining a coarser, shape-regular macro-mesh  $\mathcal{T}_m$ . Theorem 12 is the main tool of our macro-element technique.

First, we introduce local bilinear forms. If  $M \in \mathcal{T}_m$ , we define

$$B_{h,M}(\mathbf{v}, q) = - \int_M q \nabla_h \cdot \mathbf{v} \, d\mathbf{x} + \int_{\mathcal{E}_T \cap M} \{\{q\}\} \llbracket \mathbf{v} \rrbracket \, ds + \int_{\mathcal{E} \cap \partial M} q \mathbf{v} \cdot \mathbf{n} \, ds, \quad (6.1)$$

for  $(\mathbf{v}, q) \in \mathbf{V}_h^k(\mathcal{T}) \times Q_h^{k-1}(\mathcal{T})$ . Correspondingly, we also need the local norm

$$\|\mathbf{v}\|_{h,M}^2 = \sum_{K \in \mathcal{T}, K \subset M} |\mathbf{v}|_{1,K}^2 + \int_{\mathcal{E}_T \cap M} \delta_M |\llbracket \mathbf{v} \rrbracket|^2 \, ds + \int_{\mathcal{E} \cap \partial M} \delta_M |\mathbf{v} \otimes \mathbf{n}_M|^2 \, ds, \quad (6.2)$$

where  $\mathbf{n}_M$  denotes the outward normal unit vector to  $\partial M$  and  $\delta_M$  is a discontinuity stabilization function defined as in (3.3), with  $h(\mathbf{x})$  replaced by

$$h_M(\mathbf{x}) := \begin{cases} h(\mathbf{x}) & \mathbf{x} \in f \subset \mathcal{E}_T \setminus \partial M, \\ h_f & \mathbf{x} \in f \subset \partial M. \end{cases} \quad (6.3)$$

By integration by parts on each element in  $M$ , we have

$$B_{h,M}(\mathbf{v}, q) = \int_M \mathbf{v} \cdot \nabla_h q \, d\mathbf{x} - \int_{\mathcal{E}_T \cap M} \llbracket q \rrbracket \cdot \{\{ \mathbf{v} \} \} \, ds. \quad (6.4)$$

If  $\mathcal{T}_M$  is the restriction of  $\mathcal{T}$  to  $M$ , then

$$B_{h,M}(\mathbf{v}, q) = B_h(\mathbf{v}, q), \quad \mathbf{v} \in \tilde{\mathbf{V}}_h^k(\mathcal{T}_M; M), \quad (6.5)$$

where we use the same notation for  $\mathbf{v} \in \tilde{\mathbf{V}}_h^k(\mathcal{T}_M; M)$  and its extension by zero to  $\Omega$ .

For a geometric edge mesh on  $\Omega$ , we have

$$\delta(\mathbf{x}) \leq c\delta_M(\mathbf{x}), \quad \delta(\mathbf{x}) \leq c\delta_{M'}(\mathbf{x}), \quad \mathbf{x} \in \partial M \cap \partial M', \quad (6.6)$$

with  $c > 0$  solely depending on  $\sigma$  and the shape-regularity of the macro-element mesh  $\mathcal{T}_m$ . This follows from the construction of geometric edge meshes, from the definition of  $\delta$  in (3.2), (3.3), and from (3.1).

The following theorem holds.

**THEOREM 12** Let  $\mathcal{T} = \mathcal{T}^{n,\sigma}$  be a geometric edge mesh on  $\Omega$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Let  $\mathcal{T}_m$  be the underlying macro-element mesh. Assume that there exists a low-order space  $\mathbf{X}_h \subseteq \mathbf{V}_h^k(\mathcal{T})$  such that

$$\inf_{0 \neq q \in Q_h^0(\mathcal{T}_m)} \sup_{0 \neq \mathbf{v} \in \mathbf{X}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_0} \geq C_1, \quad (6.7)$$

with a constant  $C_1 > 0$  independent of  $k$ . Furthermore, assume that there exists a constant  $C_2 > 0$  independent of  $M \in \mathcal{T}_m$  and  $k$  such that

$$\inf_{0 \neq q \in Q_h^{k-1}(\mathcal{T}_M; M)} \sup_{0 \neq \mathbf{v} \in \tilde{\mathbf{V}}_h^k(\mathcal{T}_M; M)} \frac{B_{h,M}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,M} \|q\|_{0,M}} \geq C_2 k^{-\alpha}, \quad M \in \mathcal{T}_m, \quad (6.8)$$

with  $\alpha \geq 0$  and  $\mathcal{T}_M$  denoting the restriction of  $\mathcal{T}$  to  $M \in \mathcal{T}_m$ . Then the spaces  $\mathbf{V}_h^k(\mathcal{T})$  and  $Q_h^{k-1}(\mathcal{T})$  satisfy

$$\inf_{0 \neq q \in Q_h^{k-1}(\mathcal{T})} \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h^k(\mathcal{T})} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_0} \geq C k^{-\alpha},$$

with a constant  $C > 0$  solely depending on  $C_1, C_2, \sigma$  and the shape-regularity of  $\mathcal{T}_m$ .

*Proof.* Let  $q \in Q_h^{k-1}(\mathcal{T})$ . We decompose  $q$  into  $q = q^* + q_m$  where  $q_m$  is the  $L^2(\Omega)$ -projection of  $q$  onto the space  $Q_h^0(\mathcal{T}_m)$  of piecewise constant pressures on the macro-element mesh  $\mathcal{T}_m$ . Because of (6.7), there exists  $\mathbf{v}_m \in \mathbf{X}_h$  such that

$$B_h(\mathbf{v}_m, q_m) \geq \|q_m\|_0^2, \quad \|\mathbf{v}_m\|_h \leq C_1^{-1} \|q_m\|_0. \quad (6.9)$$

We next consider  $q^* \in Q_h^{k-1}(\mathcal{T})$ . We fix a macro-element  $M \in \mathcal{T}_m$  and set  $q_M^* := q^*|_M$ . We note that  $q_M^*$  has vanishing mean value on  $M$ . By using (6.8), there exists a field  $\mathbf{v}_M^*$  in  $\tilde{\mathbf{V}}_h^k(\mathcal{T}_M; M)$  such that

$$B_{h,M}(\mathbf{v}_M^*, q_M^*) \geq \|q_M^*\|_{0,M}^2, \quad \|\mathbf{v}_M^*\|_{h,M} \leq C_2^{-1} k^\alpha \|q_M^*\|_{0,M}. \quad (6.10)$$

We now define  $\mathbf{v}^* = \sum_{M \in \mathcal{T}_m} \mathbf{v}_M^*$ . By construction,  $\mathbf{v}_M^*$  has a vanishing normal component on  $\partial M$  and vanishes outside  $M$ . Thus, combining (6.5) with (6.10) yields

$$B_h(\mathbf{v}^*, q^*) = \sum_{M \in \mathcal{T}_m} B_{h,M}(\mathbf{v}_M^*, q_M^*) \geq \|q^*\|_0^2. \quad (6.11)$$

Furthermore, thanks to (6.6) and (6.10),

$$\|\mathbf{v}^*\|_h^2 \leq C \sum_{M \in \mathcal{T}_m} \|\mathbf{v}_M^*\|_{h,M}^2 \leq C k^{2\alpha} \|q^*\|_0^2, \quad (6.12)$$

with a constant  $C$  only depending on  $C_2$  and the constant in (6.6). Select now  $\mathbf{v} = \mathbf{v}_m + \eta \mathbf{v}^* \in \mathbf{V}_h^k(T)$  where  $\eta > 0$  is still at our disposal. First, thanks to (6.5), (6.4) and the fact that  $q_m$  is constant on each macro-element, we have

$$\begin{aligned} B_h(\mathbf{v}, q_m) &= \sum_{M \in \mathcal{T}_m} B_{h,M}(\mathbf{v}_M^*, q_m) \\ &= \sum_{M \in \mathcal{T}_m} \left( \int_M \mathbf{v}_M^* \cdot \nabla_h q_m \, d\mathbf{x} - \int_{\mathcal{E}_T \cap M} \llbracket q_m \rrbracket \cdot \{\{\mathbf{v}_M^*\}\} \, ds \right) = 0. \end{aligned}$$

Further, the continuity of  $B_h(\cdot, \cdot)$  in Theorem 5, (6.9), and the arithmetic–geometric mean inequality yield

$$|B_h(\mathbf{v}_m, q^*)| \leq \alpha_2 \|\mathbf{v}_m\|_h \|q^*\|_0 \leq C \|q_m\|_0 \|q^*\|_0 \leq \frac{C}{\varepsilon} \|q_m\|_0^2 + \varepsilon C \|q^*\|_0^2,$$

with another parameter  $\varepsilon > 0$  to be properly chosen. Combining the above results with (6.9) and (6.11), gives

$$\begin{aligned} B_h(\mathbf{v}, q) &= B_h(\mathbf{v}_m, q_m) + B_h(\mathbf{v}_m, q^*) + \eta B_h(\mathbf{v}^*, q^*) \\ &\geq \left(1 - \frac{C}{\varepsilon}\right) \|q_m\|_0^2 + (\eta - \varepsilon C) \|q^*\|_0^2. \end{aligned}$$

It is then clear that we can choose  $\eta$  and  $\varepsilon$  in such a way that

$$B_h(\mathbf{v}, q) \geq c \|q\|_0^2 \quad (6.13)$$

with a constant  $c$  independent of  $k$ . Furthermore, from (6.9) and (6.12),

$$\|\mathbf{v}\|_h \leq \|\mathbf{v}_m\|_h + \eta \|\mathbf{v}^*\|_h \leq c k^\alpha \|q\|_0. \quad (6.14)$$

The assertion of Theorem 12 follows then from (6.13) and (6.14).  $\square$

For geometric edge meshes, the macro-elements are refined by mapping reference configurations on  $\widehat{Q}$ . Condition (6.8) in Theorem 12 can then be verified by checking the stability of the patches on the reference cube  $\widehat{Q}$ . Similarly to (6.1) and (6.2), we denote by  $B_{h,\widehat{Q}}(\cdot, \cdot)$  and  $\|\cdot\|_{h,\widehat{Q}}$  the divergence form and the broken energy norm on a mesh on  $\widehat{Q}$ , respectively, with the stabilization function  $\delta_{\widehat{Q}}$  defined according to (3.3), but with  $h$  replaced by the local mesh size  $h_{\widehat{Q}}$  defined as in (6.3) with  $M = \widehat{Q}$ .



PROPOSITION 13 Let  $\mathcal{T} = \mathcal{T}^{n,\sigma}$  be a geometric edge mesh on  $\Omega$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Let  $\mathcal{T}_m$  be the underlying macro-element mesh, and  $\mathcal{F}$  be a family of meshes on the reference element  $\widehat{Q}$ , also containing the trivial triangulation  $\widehat{\mathcal{T}} = \{\widehat{Q}\}$ . Assume that  $\mathcal{T}$  is obtained from  $\mathcal{T}_m$  by further partitioning the elements of  $\mathcal{T}_m$  into  $F_M(\widehat{\mathcal{T}})$  where  $\widehat{\mathcal{T}} \in \mathcal{F}$  and  $F_M$  is the affine mapping between  $\widehat{Q}$  and  $M$ . Assume that the family  $\mathcal{F}$  is uniformly stable in the sense that

$$\inf_{0 \neq q \in Q_h^{k-1}(\widehat{\mathcal{T}}; \widehat{Q})} \sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\widehat{\mathcal{T}}; \widehat{Q})} \frac{B_{h,\widehat{Q}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,\widehat{Q}} \|q\|_{0,\widehat{Q}}} \geq C k^{-\alpha}, \quad \widehat{\mathcal{T}} \in \mathcal{F}, \quad k \geq 1, \quad (6.15)$$

with a constant  $C > 0$  independent of  $\widehat{\mathcal{T}} \in \mathcal{F}$  and  $k$ . Then, condition (6.8) in Theorem 12 is satisfied with a constant that only depends on the constant in (6.15) and the shape-regularity of the macro-element mesh  $\mathcal{T}_m$ .

*Proof.* Let  $M \in \mathcal{T}_m$  be a macro-element. The restriction  $\mathcal{T}_M$  of  $\mathcal{T}$  to  $M$  is given by  $F_M(\widehat{\mathcal{T}})$  for some mesh  $\widehat{\mathcal{T}} \in \mathcal{F}$ . Let  $q \in Q_h^{k-1}(\mathcal{T}_M; M)$ . We transform  $q$  back to the reference element  $\widehat{Q}$  via the affine transformation  $F_M : \widehat{Q} \rightarrow M$ : that is, we set  $\widehat{q} = q \circ F_M \in Q_h^{k-1}(\widehat{\mathcal{T}}; \widehat{Q})$ . By (6.15), there exists  $\widehat{\mathbf{v}} \in \widetilde{\mathbf{V}}_h^k(\widehat{\mathcal{T}}; \widehat{Q})$  such that

$$B_{h,\widehat{Q}}(\widehat{\mathbf{v}}, \widehat{q}) \geq \|\widehat{q}\|_{0,\widehat{Q}}^2, \quad \|\widehat{\mathbf{v}}\|_{h,\widehat{Q}} \leq C^{-1} k^\alpha \|\widehat{q}\|_{0,\widehat{Q}}. \quad (6.16)$$

We use the Piola transform, see Brezzi & Fortin (1991, Section III.1), and set

$$\mathbf{v} = P_M(\widehat{\mathbf{v}}) = |J_M|^{-1} J_M \widehat{\mathbf{v}} \circ F_M^{-1}.$$

Here,  $J_M$  is the Jacobian of  $F_M$  and  $|J_M| = |\det(J_M)|$ . Let now  $K = F_M(\tilde{K})$  be an element of  $M$  that is the image of the element  $\tilde{K}$  in  $\widehat{Q}$ . It can then be easily seen that  $\mathbf{v}|_K$  is obtained from  $\widehat{\mathbf{v}}|_{\tilde{K}}$  through the local Piola transformation  $\tilde{K} \rightarrow K$ . Due to the properties of these transforms in Brezzi & Fortin (1991, Lemmas 1.5 and 1.6), we thus have  $\mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_M; M)$  and  $B_{h,\widehat{Q}}(\widehat{\mathbf{v}}, \widehat{q}) = B_{h,M}(\mathbf{v}, q)$ . By using the definition of  $\delta_M$  and  $\delta_{\widehat{Q}}$  and standard scaling properties for the Piola transform, we obtain from (6.16) the existence of a field in  $\widetilde{\mathbf{V}}_h^k(\mathcal{T}_M; M)$  also denoted by  $\mathbf{v}$  such that

$$B_{h,M}(\mathbf{v}, q) \geq \|q\|_{0,M}^2, \quad \|\mathbf{v}\|_{h,M} \leq C k^\alpha \|q\|_{0,M},$$

where  $C$  solely depends on the constant in (6.15) and the shape-regularity of the macro-element mesh  $\mathcal{T}_m$ .  $\square$

REMARK 14 The condition that the patch maps be affine may be weakened to the extent that the meshes are patchwise mapped from suitable reference patches by smooth, bijective and nondegenerate maps. In this case, the macro-element technique can be modified as in Chilton & Suri (2000) which requires suitably adapted velocity spaces in the physical coordinates.

## 7. Raviart–Thomas interpolant on anisotropic meshes

The purpose of this section is to provide estimates for the interpolant on Raviart–Thomas finite-element spaces on certain anisotropic meshes. In order to do so, we employ a

different representation than that considered in Schötzau *et al.* (2003), which was originally proposed in Ainsworth & Pinchedez (2002). The representation here was first proposed and proven in Hientzsch (2001); see in particular Chapter 7. Here we propose a simpler proof.

### 7.1 One-dimensional interpolants

We first introduce some one-dimensional projections. Let  $\{L_i(x), i \in \mathbb{N}_0\}$  be the set of orthogonal Legendre polynomials on  $\hat{I} = (-1, 1)$ ; see e.g. Bernardi & Maday (1997). We also consider a different set  $\{U_i(x), i \in \mathbb{N}_0\}$ :

$$\begin{aligned} U_0(x) &= L_0(x) = 1, & U_1(x) &= L_1(x) = x, \\ U_i(x) &= \int_{-1}^x L_{i-1}(t) dt = (2i-1)^{-1}(L_i - L_{i-2}), & i &\geq 2; \end{aligned} \quad (7.1)$$

see in particular Theorem 3.3 of Bernardi & Maday (1997).

For a generic interval  $I = (x_1, x_2) = F_I(\hat{I})$ , two bases can be found by mapping  $\{L_i\}$  and  $\{U_i\}$  onto  $I$ . In the following, we use the same notation for these bases in  $L^2(I)$  as for the reference interval.

Let  $\pi_k^0 : L^2(I) \rightarrow \mathbb{Q}_k(I)$  be the  $L^2$ -orthogonal projection. We note that

$$\pi_k^0 \left( \sum_{i=0}^{\infty} v_i L_i \right) = \sum_{i=0}^k v_i L_i.$$

We also define a second projection  $\pi_k^1 : L^2(I) \rightarrow \mathbb{Q}_k(I)$  by

$$\pi_k^1 \left( \sum_{i=0}^{\infty} \tilde{v}_i U_i \right) = \sum_{i=0}^k \tilde{v}_i U_i.$$

LEMMA 15 Let  $I = (x_1, x_2)$ . For  $v \in H^1(I)$ , we have

$$\begin{aligned} (\pi_k^1 v)(x_1) &= v(x_1), & (\pi_k^1 v)(x_2) &= v(x_2), & k &\geq 1, \\ \int_I \pi_k^1 v q dx &= \int_I v q dx, & q &\in \mathbb{Q}_{k-2}(I), & k &\geq 2. \end{aligned}$$

*Proof.* The first property follows from the fact that  $U_i(x_1) = U_i(x_2) = 0$  for  $i \geq 2$ . To prove the second property, let  $q \in \mathbb{Q}_{k-2}(I)$  be given by  $q = L'_{i-1}$  for  $2 \leq i \leq k$ . It is then easy to see that

$$\int_I (\pi_k^1 v)' L_{i-1} dx = \int_I v' L_{i-1} dx.$$

From the above identity and the first assertion, the second assertion follows by integration by parts.  $\square$

The next lemma provides certain stability estimates.

LEMMA 16 Let  $I = (x_1, x_2)$  and  $v \in H^1(I)$ . There is a constant  $C > 0$  independent of  $k$  and  $I$  such that

$$\|\pi_k^0 v\|_{0,I} \leq \|v\|_{0,I}, \quad |\pi_k^0 v|_{1,I} \leq C\sqrt{k} |v|_{1,I}, \quad |\pi_k^1 v|_{1,I} \leq |v|_{1,I}.$$

If in addition  $v \in H_0^1(I)$ , then

$$\|\pi_k^1 v\|_{0,I} \leq C\sqrt{k} \|v\|_{0,I}. \quad (7.2)$$

*Proof.* Since for a generic interval the bounds are obtained by a standard scaling argument, it is enough to consider  $I = (-1, 1)$ . The bounds for  $\pi_k^0$  can be found in Canuto & Quarteroni (1982). Moreover, let  $v = \sum_{i=0}^{\infty} v_i U_i$  and  $\chi : [0, \infty) \rightarrow \mathbb{R}$  be a  $C^1$  cut-off function that is equal to one in  $[0, 1]$ , decreases to zero in  $[1, 1 + \mu]$ , and is equal to zero in  $[1 + \mu, \infty)$ . If  $\mu = 1/k$ , it is easy to prove that  $\pi_k^1 v = \sum_{i=0}^{\infty} \chi\left(\frac{i}{k}\right) v_i U_i$ . The bounds for  $\pi_k^1$  can then be found in Bernardi & Maday (1999, Lemmas 3.2 and 3.3, and Remark 3.4).  $\square$

Further, we will make use of the following approximation property. It is proved in Houston *et al.* (2002) for the reference interval and can be proved for a generic interval by a scaling argument.

LEMMA 17 Let  $I = (x_1, x_2)$  and  $h = x_2 - x_1$ . Then there is a constant  $C > 0$  independent of  $k$  and  $I$  such that for  $v \in H^1(I)$

$$|(\pi_k^0 v - v)(x_i)|^2 \leq C \frac{h}{k} |v|_{1,I}^2, \quad i = 1, 2.$$

## 7.2 Two-dimensional interpolants

We recall some two-dimensional results that were proven in Ainsworth & Pinchedez (2002) and Schötzau *et al.* (2003). Given the reference square  $\widehat{S}$  and an integer  $k \geq 0$ , we consider the Raviart–Thomas space

$$RT_k(\widehat{S}) = \mathbb{Q}_{k+1,k}(\widehat{S}) \times \mathbb{Q}_{k,k+1}(\widehat{S}),$$

where  $\mathbb{Q}_{k_1,k_2}(\widehat{S})$  is the space of polynomials of degree  $k_i$  in the  $i$ th variable on  $\widehat{S}$ . For an affinely mapped element  $K = F_K(\widehat{S})$ , the Raviart–Thomas space  $RT_k(K)$  is defined by suitably mapping functions in  $RT_k(\widehat{S})$  using a Piola transformation; see Brezzi & Fortin (1991, Section 3.3) or Ainsworth & Pinchedez (2002, Section 3.3) for further details.

On  $\widehat{S}$ , there is a unique interpolation operator  $\Pi_{\widehat{S}}^k = \Pi_{\widehat{S}}^k : H^1(\widehat{S})^2 \rightarrow RT_k(\widehat{S})$ , such that

$$\begin{aligned} \int_{\widehat{S}} (\Pi_{\widehat{S}}^k \mathbf{v} - \mathbf{v}) \cdot \mathbf{w} \, d\mathbf{x} &= 0 \quad \forall \mathbf{w} \in \mathbb{Q}_{k-1,k}(\widehat{S}) \times \mathbb{Q}_{k,k-1}(\widehat{S}), \\ \int_{\widehat{f}_m} (\Pi_{\widehat{S}}^k \mathbf{v} - \mathbf{v}) \cdot \mathbf{n} \, d\mathbf{s} &= 0 \quad \forall \varphi \in \mathbb{Q}_k(\widehat{f}_m), \quad m = 1, \dots, 4; \end{aligned} \quad (7.3)$$

see Brezzi & Fortin (1991) or Ainsworth & Pinchedez (2002). For  $k = 0$ , the first condition in (7.3) is void. For an affinely mapped element  $K$ , the interpolant  $\Pi_K = \Pi_K^k : H^1(K)^2 \rightarrow RT_k(K)$  can be defined by using a Piola transform in such a way that the orthogonality conditions in (7.3) also hold for  $\Pi_K$ .

For shape-regular elements, we recall the following result from Schötzau *et al.* (2003, Lemma 6.9 and 6.10).

LEMMA 18 Let  $K$  be a shape-regular element of diameter  $h_K$  and  $\mathbf{v} \in H^1(K)^2$ . Then

$$|\Pi_K \mathbf{v}|_{1,K} \leq C k |\mathbf{v}|_{1,K}, \quad \|\mathbf{v} - \Pi_K \mathbf{v}\|_{0,\partial K}^2 \leq C h_K |\mathbf{v}|_{1,K}^2,$$

with a constant  $C > 0$  that is independent of  $k$  and  $h_K$ .

In addition to the bounds in Lemma 18, we need slightly refined estimates to treat axiparallel elements of the form  $S = S_{xy} = (x_1, x_2) \times (y_1, y_2)$ . Such bounds can be obtained by using tensor product arguments. For this purpose, we define the two-dimensional operators

$$\Pi_k^x := \pi_k^{0,y} \circ \pi_{k+1}^{1,x}, \quad \Pi_k^y := \pi_{k+1}^{1,y} \circ \pi_k^{0,x},$$

with the one-dimensional projectors  $\pi_k^0$  and  $\pi_k^1$  from Section 7.1. We have specified the variable on which these projections act.

We have the following representation result; see also Section 7.6.1 and formula (7.17) in Hientzsch (2001).

LEMMA 19 The Raviart–Thomas projector on  $S = S_{xy} = (x_1, x_2) \times (y_1, y_2)$  satisfies

$$\Pi_S^k \mathbf{v} = \Pi_S^k(v_x, v_y) = (\Pi_k^x v_x, \Pi_k^y v_y), \quad \mathbf{v} \in C^\infty(\bar{S})^2.$$

*Proof.* Using Lemma 15 and properties of the  $L^2$ -projection, it is immediate to see that  $(\Pi_k^x v_x, \Pi_k^y v_y)$  satisfies the conditions in (7.3) on  $S$ .  $\square$

The operators  $\Pi_k^x$  and  $\Pi_k^y$  can be uniquely extended by density to functions in  $H^1(S)$  (these extensions being still denoted by  $\Pi_k^x$  and  $\Pi_k^y$ ). This is a consequence of the following result.

LEMMA 20 Let  $v \in C^\infty(\bar{S})$ . Then there exists a constant  $C$  independent of  $k$ , such that

$$\|\partial_x(\Pi_k^x v)\|_{0,\hat{S}} \leq \|\partial_x v\|_{0,\hat{S}}, \quad \|\partial_y(\Pi_k^x v)\|_{0,\hat{S}} \leq C k |v|_{1,\hat{S}}.$$

Similar estimates hold for  $\Pi_k^y$ .

*Proof.* The first bound can be proven using the definition of  $\Pi_k^x$  and  $\Pi_k^y$  and the one-dimensional bounds in Lemma 16. The second bound can be found in Schötzau *et al.* (2003, Lemma 6.9).  $\square$

We end this section with an error estimate for the two-dimensional  $L^2$ -projection. It can be proven by using Lemma 17; cf. Lemma 3.9 of Houston *et al.* (2002).

LEMMA 21 Let  $S = S_{xy} = (x_1, x_2) \times (y_1, y_2)$  be a shape-regular element of diameter  $h$ . Then there exists a constant  $C > 0$  independent of  $k$  and  $h$  such that

$$\|v - \pi_k^{0,y} \pi_k^{0,x} v\|_{0,\partial S}^2 \leq C \frac{h}{k} |v|_{1,S}^2, \quad v \in H^1(S).$$

### 7.3 Three-dimensional interpolants

In this section, we introduce the Raviart–Thomas interpolant in three dimensions. We use the same notation as for the two-dimensional case. Given an axiparallel element of the form

$$K_{xyz} = (x_1, x_2) \times (y_1, y_2) \times (z_1, z_2),$$

and an integer  $k \geq 0$ , we consider the Raviart–Thomas space

$$RT_k(K_{xyz}) = \mathbb{Q}_{k+1,k,k}(K_{xyz}) \times \mathbb{Q}_{k,k+1,k}(K_{xyz}) \times \mathbb{Q}_{k,k,k+1}(K_{xyz}),$$

where  $\mathbb{Q}_{k_1,k_2,k_3}(K_{xyz})$  is the space of polynomials of degree  $k_i$  in the  $i$ th variable on  $K_{xyz}$ . For general affinely mapped elements  $K \in \mathcal{T}$  of a geometric edge mesh  $\mathcal{T} = \mathcal{T}^{n,\sigma}$  (see Property 2), the Raviart–Thomas space  $RT_k(K)$  is defined by suitably mapping functions in  $RT_k(K_{xyz})$  using a Piola transformation; see Brezzi & Fortin (1991) or Ainsworth & Pinchedez (2002) for further details.

On  $K_{xyz}$ , there is a unique interpolation operator  $\Pi_{K_{xyz}} = \Pi_{K_{xyz}}^k : H^1(K_{xyz})^3 \rightarrow RT_k(K_{xyz})$ , such that

$$\begin{aligned} \int_{K_{xyz}} (\Pi_{K_{xyz}} \mathbf{v} - \mathbf{v}) \cdot \mathbf{w} \, d\mathbf{x} &= 0 \\ \forall \mathbf{w} \in \mathbb{Q}_{k-1,k,k}(K_{xyz}) \times \mathbb{Q}_{k,k-1,k}(K_{xyz}) \times \mathbb{Q}_{k,k,k-1}(K_{xyz}), \quad (7.4) \\ \int_{f_m} (\Pi_{K_{xyz}} \mathbf{v} - \mathbf{v}) \cdot \mathbf{n} \, d\mathbf{s} &= 0 \quad \forall \varphi \in \mathbb{Q}_{k,k}(f_m), \quad m = 1, \dots, 6; \end{aligned}$$

with  $\{f_m\}$  denoting the six faces of  $K_{xyz}$ , see Brezzi & Fortin (1991) or Ainsworth & Pinchedez (2002). For  $k = 0$ , the first condition in (7.4) is void. For an element  $K \in \mathcal{T}$ , the interpolant  $\Pi_K = \Pi_K^k : H^1(K)^3 \rightarrow RT_k(K)$  can be defined by using a Piola transform in such a way that the orthogonality conditions in (7.4) also hold for  $\Pi_K$ .

We now define the three-dimensional operators on  $K = K_{xyz}$

$$\Pi_k^x := \pi_k^{0,z} \circ \pi_k^{0,y} \circ \pi_{k+1}^{1,x}, \quad \Pi_k^y := \pi_k^{0,z} \circ \pi_{k+1}^{1,y} \circ \pi_k^{0,x}, \quad \Pi_k^z := \pi_{k+1}^{1,z} \circ \pi_k^{0,y} \circ \pi_k^{0,x},$$

where we have specified the variable on which the one-dimensional projections act. The following representation result can be proven in the same way as in two dimensions; see also Section 7.6.2 and formula (7.19) in Hientzsch (2001).

LEMMA 22 On  $K = K_{xyz}$ , the Raviart–Thomas interpolant satisfies

$$\Pi_K^k \mathbf{v} = \Pi_K^k(v_x, v_y, v_z) = (\Pi_k^x v_x, \Pi_k^y v_y, \Pi_k^z v_z), \quad \mathbf{v} \in C^\infty(\overline{K}).$$

The operators  $\Pi_k^x$ ,  $\Pi_k^y$ , and  $\Pi_k^z$  are well-defined for functions in  $C^\infty(\overline{K})$  and can be uniquely extended by density to  $H^1(K)$  (these extensions being still denoted by  $\Pi_k^x$ ,  $\Pi_k^y$  and  $\Pi_k^z$ ). This is a consequence of the following result.

LEMMA 23 Let  $v \in C^\infty(\overline{\widehat{Q}})$ . Then there exists a constant  $C$  independent of  $k$  such that

$$\begin{aligned}\|\partial_x(\Pi_k^x v)\|_{0,\widehat{Q}}^2 &\leq C \|\partial_x v\|_{0,\widehat{Q}}^2, \\ \|\partial_y(\Pi_k^x v)\|_{0,\widehat{Q}}^2 &\leq Ck^2 (\|\partial_y v\|_{0,\widehat{Q}}^2 + \|\partial_x v\|_{0,\widehat{Q}}^2), \\ \|\partial_z(\Pi_k^x v)\|_{0,\widehat{Q}}^2 &\leq Ck^2 (\|\partial_z v\|_{0,\widehat{Q}}^2 + \|\partial_x v\|_{0,\widehat{Q}}^2).\end{aligned}$$

Similar estimates hold for  $\Pi_k^y$  and  $\Pi_k^z$ .

*Proof.* The first two estimates can be obtained using Lemmas 16 and 20, and the fact that  $\Pi_k^x$  can be written as the tensor product of the two-dimensional Raviart–Thomas projection and a one-dimensional  $L^2$ -projection:  $\Pi_k^x = \pi_k^{0,z} \circ (\pi_k^{0,y} \circ \pi_k^{1,x})$ ; see Lemma 22. The last bound can be obtained by exchanging the roles of the  $y$  and  $z$  variables.  $\square$

#### 7.4 Stretched elements

For a general anisotropic element, Lemma 23 and a scaling argument provide estimates that are not independent of the aspect ratio. For an edge patch on  $\widehat{Q}$ , however, we only need to consider stretched elements of the form

$$K_{xyz} = (x_1, x_2) \times (y_1, y_2) \times \widehat{I}, \quad (7.5)$$

with  $h_x = x_2 - x_1 < 2$ ,  $h_y = y_2 - y_1 < 2$ , and  $h_x$  comparable to  $h_y$ . Even for this simpler case, good bounds cannot be found for all the components. However, if we only consider vectors with a vanishing normal component along the faces  $z = \pm 1$ , we have the following result.

LEMMA 24 Let  $K$  be given by (7.5) and  $\mathbf{v} = (v_x, v_y, v_z) \in H^1(K)^3$ , such that  $\mathbf{v} \cdot \mathbf{n}_\pm = 0$  along  $z = \pm 1$ , with  $\mathbf{n}_\pm = (0, 0, \pm 1)$ . If  $ch_x \leq h_y \leq Ch_x$ , then there exists a constant independent of  $k$  and the aspect ratio of  $K$ , such that

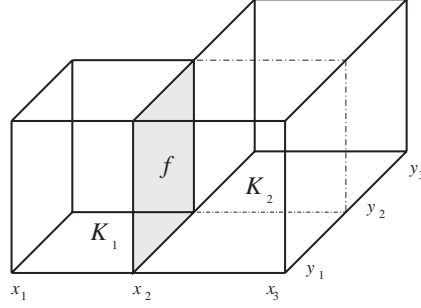
$$\begin{aligned}\|\partial_x(\Pi_k^x v_x)\|_{0,K}^2 &\leq C \|\partial_x v_x\|_{0,K}^2, \\ \|\partial_y(\Pi_k^x v_x)\|_{0,K}^2 &\leq Ck^2 (\|\partial_y v_x\|_{0,K}^2 + \|\partial_x v_x\|_{0,K}^2), \\ \|\partial_z(\Pi_k^x v_x)\|_{0,K}^2 &\leq Ck^2 (\|\partial_z v_x\|_{0,K}^2 + \|\partial_x v_x\|_{0,K}^2),\end{aligned}$$

and similarly for  $\Pi_k^y v_y$ . In addition,

$$\begin{aligned}\|\partial_x(\Pi_k^z v_z)\|_{0,K}^2 &\leq Ck^2 \|\partial_x v_z\|_{0,K}^2, \\ \|\partial_y(\Pi_k^z v_z)\|_{0,K}^2 &\leq Ck^2 \|\partial_y v_z\|_{0,K}^2, \\ \|\partial_z(\Pi_k^z v_z)\|_{0,K}^2 &\leq C \|\partial_z v_z\|_{0,K}^2.\end{aligned}$$

Consequently,  $|\Pi_K \mathbf{v}|_{1,K} \leq Ck |\mathbf{v}|_{1,K}$ , with a constant independent of  $k$  and the aspect ratio of  $K$ .

*Proof.* Assume first that  $\mathbf{v} \in C^\infty(\overline{K})^3$ . The bounds for  $\Pi_k^x v_x$  and  $\Pi_k^y v_y$  follow from Lemma 23 and a scaling argument. To obtain the estimates of  $\Pi_k^z v_z$ , we use the representation in Lemma 22 and the results in Lemma 16. In particular, we use (7.2) to bound  $\pi_{k+1}^{1,z}$ . The proof is then completed by a density argument.  $\square$

FIG. 2. Two stretched elements  $K_1$  and  $K_2$  that share the face  $f = \{x_2\} \times (y_1, y_2) \times \widehat{I}$ .

Similarly, it is possible to bound the jumps across faces of stretched elements. Let  $K_1$  and  $K_2$  be two stretched elements given by

$$K_1 = (x_1, x_2) \times (y_1, y_2) \times \widehat{I}, \quad K_2 = (x_2, x_3) \times (y_1, y_3) \times \widehat{I}, \quad (7.6)$$

with  $y_2 \leq y_3$ . Further, we introduce the faces  $f_1 = \{x_2\} \times (y_1, y_2) \times \widehat{I}$  and  $f_2 = \{x_2\} \times (y_1, y_3) \times \widehat{I}$ . Let  $f = f_1 \subseteq f_2$ , as illustrated in Fig. 2. We then set  $h_{1,x} = x_2 - x_1$ ,  $h_{2,x} = x_3 - x_2$ ,  $h_{1,y} = y_2 - y_1$ , and  $h_{2,y} = y_3 - y_1$ .

LEMMA 25 Let  $K_1$  and  $K_2$  be the two stretched elements in (7.6). Let  $\mathbf{u} \in H^1(K_1 \cup K_2)^3$  such that  $\mathbf{u} \cdot \mathbf{n}_\pm = 0$  along  $z = \pm 1$ , with  $\mathbf{n}_\pm = (0, 0, \pm 1)$ . Assume that

$$ch_{1,x} \leq h_{2,x} \leq Ch_{1,x}, \quad h_{1,y} \leq h_{2,y} \leq Ch_{2,x}.$$

Let  $\mathbf{v}$  be the piecewise polynomial given by  $\mathbf{v}|_{K_i} = \Pi_{K_i}(\mathbf{u}|_{K_i})$  where  $\Pi_{K_i}$  is the Raviart-Thomas projector of degree  $k$  on  $K_i$ ,  $i = 1, 2$ . Then,

$$\int_f \|\llbracket \mathbf{v} \rrbracket\|^2 ds \leq Ch_{1,x} [\|\partial_x \mathbf{u}\|_{0,K_1}^2 + \|\partial_y \mathbf{u}\|_{0,K_1}^2] + Ch_{2,x} [\|\partial_x \mathbf{u}\|_{0,K_2}^2 + \|\partial_y \mathbf{u}\|_{0,K_2}^2],$$

with a constant  $C > 0$  that is independent of  $k$  and the mesh sizes  $h_{1,x}$ ,  $h_{2,x}$ ,  $h_{1,y}$ , and  $h_{2,y}$ .

*Proof.* First, we assume that  $\mathbf{u} \in C^\infty(\overline{K_1} \cup \overline{K_2})^3$ .

For  $i = 1, 2$ , we denote  $\mathbf{u}|_{K_i}$  by  $\mathbf{u}^i = (u_x^i, u_y^i, u_z^i)$  and  $\mathbf{v}|_{K_i}$  by  $\mathbf{v}^i = (v_x^i, v_y^i, v_z^i)$ . Since

$$\int_f \|\llbracket \mathbf{v} \rrbracket\|^2 ds = \int_f (v_x^1 - v_x^2)^2 ds + \int_f (v_y^1 - v_y^2)^2 ds + \int_f (v_z^1 - v_z^2)^2 ds =: T_1 + T_2 + T_3,$$

it is enough to estimate the terms  $T_1$ ,  $T_2$  and  $T_3$  separately. We observe that  $v_x^1 = v_x^2$  (and thus  $T_1 = 0$ ) only if  $f = f_1 = f_2$ , since the normal component of  $\mathbf{v}$  is continuous across  $f$  in this case. In the general case, since  $u_x^1 = u_x^2$  is continuous across  $f$ , we have

$\pi_k^{0,z} u_x^1 = \pi_k^{0,z} u_x^2$  and can write

$$\begin{aligned} T_1 &= \int_f (v_x^1 - v_x^2)^2 ds \leq 2 \int_{f_1} (\pi_k^{0,z} u_x^1 - v_x^1)^2 ds + 2 \int_{f_1} (\pi_k^{0,z} u_x^2 - v_x^2)^2 ds \\ &\leq 2 \int_{f_1} (\pi_k^{0,z} u_x^1 - v_x^1)^2 ds + 2 \int_{f_2} (\pi_k^{0,z} u_x^2 - v_x^2)^2 ds := 2T_{1,A} + 2T_{1,B}. \end{aligned}$$

For  $T_{1,A}$  we use the representation in Lemma 22 of  $v_x^1 = \Pi_k^x u_x^1$  on  $K_1$ . Lemma 15 ensures

$$v_x^1 = (\pi_k^{0,z} \pi_k^{0,y} \pi_{k+1}^{1,x}) u_x^1 = (\pi_k^{0,z} \pi_k^{0,y}) u_x^1, \quad \text{on } f_1.$$

This gives

$$\begin{aligned} T_{1,A} &= \int_{f_1} (\pi_k^{0,z} u_x^1 - \pi_k^{0,z} \pi_k^{0,y} u_x^1)^2 ds \\ &\leq 2 \int_{f_1} (\pi_k^{0,z} u_x^1 - \pi_k^{0,z} \pi_k^{0,y} \pi_k^{0,x} u_x^1)^2 ds + 2 \int_{f_1} (\pi_k^{0,z} \pi_k^{0,y} (\pi_k^{0,x} u_x^1 - u_x^1))^2 ds. \end{aligned}$$

Using the stability of the  $L^2$ -projection  $\pi_k^{0,z}$  in the  $z$ -direction and the bound in Lemma 21 for  $\pi_k^{0,y} \pi_k^{0,x}$  on the shape-regular rectangle  $(x_1, x_2) \times (y_1, y_2)$  gives

$$\begin{aligned} \int_{f_1} (\pi_k^{0,z} u_x^1 - \pi_k^{0,z} \pi_k^{0,y} \pi_k^{0,x} u_x^1)^2 ds &\leq \int_{f_1} (u_x^1 - \pi_k^{0,y} \pi_k^{0,x} u_x^1)^2 ds \\ &\leq Ch_{1,x} k^{-1} [\|\partial_x u_x^1\|_{0,K_1}^2 + \|\partial_y u_x^1\|_{0,K_1}^2]. \end{aligned}$$

Similarly, using the stability of  $\pi_k^{0,z} \pi_k^{0,y}$  and the approximation result in Lemma 17 yields

$$\begin{aligned} \int_{f_1} (\pi_k^{0,z} \pi_k^{0,y} (\pi_k^{0,x} u_x^1 - u_x^1))^2 ds &\leq \int_{f_1} (\pi_k^{0,x} u_x^1 - u_x^1)^2 ds \\ &\leq Ch_{1,x} k^{-1} \|\partial_x u_x^1\|_{0,K_1}^2. \end{aligned}$$

Thus, we obtain

$$T_{1,A} \leq Ch_{1,x} k^{-1} [\|\partial_x u_x^1\|_{0,K_1}^2 + \|\partial_y u_x^1\|_{0,K_1}^2].$$

A bound for  $T_{1,B}$  can be found in the same way. Therefore,

$$T_1 \leq C \frac{h_{1,x}}{k} [\|\partial_x u_x^1\|_{0,K_1}^2 + \|\partial_y u_x^1\|_{0,K_1}^2] + C \frac{h_{2,x}}{k} [\|\partial_x u_x^2\|_{0,K_2}^2 + \|\partial_y u_x^2\|_{0,K_2}^2]. \quad (7.7)$$

Let us now consider the term  $T_2$ . Since  $u_y^1 = u_y^2$  on  $f_1$ , we have  $\pi_k^{0,z} u_y^1 = \pi_k^{0,z} u_y^2$  and can then bound  $T_2$  by

$$\begin{aligned} T_2 &= \int_f (v_y^1 - v_y^2)^2 ds \leq 2 \int_{f_1} (v_y^1 - \pi_k^{0,z} u_y^1)^2 ds + 2 \int_{f_1} (v_y^2 - \pi_k^{0,z} u_y^2)^2 ds \\ &\leq 2 \int_{f_1} (v_y^1 - \pi_k^{0,z} u_y^1)^2 ds + 2 \int_{f_2} (v_y^2 - \pi_k^{0,z} u_y^2)^2 ds =: 2T_{2,A} + 2T_{2,B}. \end{aligned}$$



Let us further estimate the term  $T_{2,A}$ . From the representation in Lemma 22 and the stability of  $\pi_k^{0,z}$  in Lemma 16, we find

$$T_{2,A} = \int_{f_1} (\pi_k^{0,z} u_y^1 - \pi_k^{0,z} \pi_{k+1}^{1,y} \pi_k^{0,x} u_y^1)^2 ds \leq \int_{f_1} (u_y^1 - \pi_{k+1}^{1,y} \pi_k^{0,x} u_y^1)^2 ds.$$

We now note that  $(\pi_{k+1}^{1,y} \pi_k^{0,x})$  is the second component of the two-dimensional Raviart–Thomas projector on the shape-regular rectangle  $(x_1, x_2) \times (y_1, y_2)$ . We can then use the two-dimensional result in Lemma 18 and obtain

$$T_{2,A} \leq Ch_{1,x} [\|\partial_x \mathbf{u}^1\|_{0,K_1}^2 + \|\partial_y \mathbf{u}^1\|_{0,K_1}^2].$$

A bound for  $T_{2,B}$  can be found in the same way. This yields

$$T_2 \leq Ch_{1,x} (\|\partial_x \mathbf{u}^1\|_{0,K_1}^2 + \|\partial_y \mathbf{u}^1\|_{0,K_1}^2) + Ch_{2,x} (\|\partial_x \mathbf{u}^2\|_{0,K_2}^2 + \|\partial_y \mathbf{u}^2\|_{0,K_2}^2). \quad (7.8)$$

For the term  $T_3$ , we note that  $u_z^1 = u_z^2$  on  $f_1$ . Thus,  $\pi_{k+1}^{1,z} u_z^1 = \pi_{k+1}^{1,z} u_z^2$  on  $f_1$  and

$$\begin{aligned} T_3 &= \int_f (v_z^1 - v_z^2)^2 ds \leq 2 \int_{f_1} (\pi_{k+1}^{1,z} u_z^1 - v_z^1)^2 ds + 2 \int_{f_1} (\pi_{k+1}^{1,z} u_z^2 - v_z^2)^2 ds \\ &\leq 2 \int_{f_1} (\pi_{k+1}^{1,z} u_z^1 - v_z^1)^2 ds + 2 \int_{f_2} (\pi_{k+1}^{1,z} u_z^2 - v_z^2)^2 ds := 2T_{3,A} + 2T_{3,B}. \end{aligned}$$

Again, we bound the two last terms separately using the representation result of Lemma 22. Since  $u_z^1$  at  $z = \pm 1$ , we also have  $\pi_k^{0,y} \pi_k^{0,x} u_z^1 = 0$  at  $z = \pm 1$ . Thus, we can use (7.2) in Lemma 16:

$$T_{3,A} = \int_{f_1} (\pi_{k+1}^{1,z} (u_z^1 - \pi_k^{0,y} \pi_k^{0,x} u_z^1))^2 ds \leq Ck \int_{f_1} (u_z^1 - \pi_k^{0,y} \pi_k^{0,x} u_z^1)^2 ds.$$

Using once more the error estimate for the  $L^2$ -projection  $\pi_k^{0,y} \pi_k^{0,x}$  on the shape-regular element  $(x_1, x_2) \times (y_1, y_2)$  in Lemma 21, we find

$$T_{3,A} \leq Ch_{1,x} [\|\partial_x u_z^1\|_{0,K_1}^2 + \|\partial_y u_z^1\|_{0,K_1}^2].$$

Since a bound for  $T_{3,B}$  can be found in the same way, we find

$$T_3 \leq Ch_{1,x} (\|\partial_x u_z^1\|_{0,K_1}^2 + \|\partial_y u_z^1\|_{0,K_1}^2) + Ch_{2,x} (\|\partial_x u_z^2\|_{0,K_2}^2 + \|\partial_y u_z^2\|_{0,K_2}^2). \quad (7.9)$$

For  $\mathbf{u} \in C^\infty(\overline{K}_1 \cup \overline{K}_2)^3$  the assertion follows by combining (7.7), (7.8) and (7.9).

The proof is extended to functions  $\mathbf{u} \in H^1(\overline{K}_1 \cup \overline{K}_2)^3$  by a density argument.  $\square$

In exactly the same manner, using the representation result of Lemma 22, we obtain the following bound for the other faces.

**LEMMA 26** Let  $K$  be an element of the form (7.5) and  $f$  an entire face of  $K$ . Assume that  $ch_x \leq h_y \leq Ch_x$ . Let  $\mathbf{u} \in H^1(K)^3$  with  $\mathbf{u}|_f = \mathbf{0}$ , and let  $\mathbf{v}$  be the Raviart–Thomas projection of  $\mathbf{u}$  of degree  $k$  on  $K$ . Then we have that

$$\int_f |\mathbf{v} \otimes \mathbf{n}_K|^2 ds \leq Ch |\mathbf{u}|_{1,K}^2,$$

with  $h = h_x \sim h_y$ . The constant  $C$  is independent of  $k$ , and the mesh sizes  $h_x$  and  $h_y$ .

*Proof.* The proof for the lateral faces parallel to the  $z$ -axis can be carried out as the proof of Lemma 25. When  $f$  is given by  $z = \pm 1$ , we can use the results in Schötzau *et al.* (2003, Lemma 6.10) for three-dimensional shape-regular elements and a scaling argument.  $\square$

### 8. Divergence stability on edge patches

Let  $\mathcal{T}_e^{n,\sigma}$  be an edge patch on  $\widehat{Q}$ . We show that  $\mathbb{Q}_k - \mathbb{Q}_{k-1}$  elements are stable on such patches with an inf-sup constant of  $\mathcal{O}(k^{-3/2})$ . The main result of this section is the following theorem.

**THEOREM 27** Let  $\mathcal{T}_e^{n,\sigma}$  be an edge patch on  $\widehat{Q}$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Let  $k \geq 1$ . Then

$$\sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})} \frac{B_{h,\widehat{Q}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,\widehat{Q}}} \geq Ck^{-3/2} \|q\|_{0,\widehat{Q}}, \quad q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q}),$$

with a constant  $C > 0$  that solely depends on  $\sigma$  and  $\delta_0$ , but is independent of  $k$ ,  $n$ , and the aspect ratio of the elements in  $\mathcal{T}_e^{n,\sigma}$ .

**REMARK 28** We emphasize that the result in Theorem 27 holds for  $k = 1$ , thus including  $\mathbb{Q}_1 - \mathbb{Q}_0$  elements. In particular, the same techniques as the ones presented here lead to a stability result of  $\mathbb{Q}_1 - \mathbb{Q}_0$  elements on irregular geometric meshes in two space dimensions. This case was not covered in Schötzau *et al.* (2003).

The proof of Theorem 27 is carried out in the next sections. We first use the results of Section 7.4, in order to prove a stability property for the Raviart–Thomas interpolant on edge patches in Corollary 29. The proof then relies on the combination of the two weaker stability results in Lemmas 31 and 32, respectively.

#### 8.1 Stability of Raviart–Thomas interpolants on edge patches

We define the Raviart–Thomas interpolant  $\Pi = \Pi^k : H^1(\widehat{Q})^3 \rightarrow \mathbf{V}_h^{k+1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$  by

$$\Pi \mathbf{u}|_K = \Pi_K^k(\mathbf{u}|_K), \quad K \in \mathcal{T}_e^{n,\sigma}. \quad (8.1)$$

We note that  $\Pi \mathbf{u}$  has a continuous normal component across elements that match regularly. If the elements match irregularly, the normal component has jumps; see, e.g. Ainsworth & Pinchedez (2002, Section 3.5). However, if  $\mathbf{u} \in H_0^1(\widehat{Q})^3$  then  $\Pi \mathbf{u}$  belongs to  $\widetilde{\mathbf{V}}_h^{k+1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$ .

We first note the following stability result.

**COROLLARY 29** Let  $\mathcal{T}_e^{n,\sigma}$  be an edge patch on  $\widehat{Q}$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. If  $\mathbf{u} \in H_0^1(\widehat{Q})^3$  and  $\Pi^k \mathbf{u}$  is the Raviart–Thomas interpolant in (8.1), then there exists a constant that solely depends on  $\sigma$  and  $\delta_0$ , but is independent of  $k$ ,  $n$ , and the aspect ratio of the elements in  $\mathcal{T}_e^{n,\sigma}$ , such that  $\|\mathbf{v}\|_{h,\widehat{Q}}^2 \leq Ck^2 |\mathbf{u}|_{1,\widehat{Q}}^2$ .

*Proof.* This follows by combining Lemma 24–26 and the definition of the penalization function  $\delta_{\widehat{Q}}$ .  $\square$

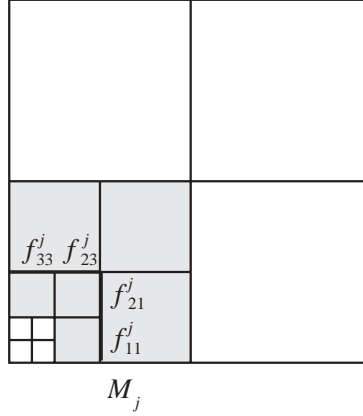


FIG. 3. Edge mesh for  $\sigma = 0.5$  and  $n = 4$ . The patch  $M_j$ ,  $j = 3$ , is the union of the shaded elements. The four interior faces  $f_{11}^j$ ,  $f_{21}^j$ ,  $f_{23}^j$  and  $f_{33}^j$  in  $M_j$  are shown in bold lines.

## 8.2 Auxiliary stability results

We establish two auxiliary stability results that we need for the proof of our main result in Theorem 27.

First we define a seminorm for the space of pressures on edge patches. We consider the interior faces of an edge patch  $\mathcal{T}_e^{n,\sigma}$  on  $\widehat{Q}$ . For  $2 \leq j \leq n$ , the patch  $M_j$  consists of six elements, the cross sections of which are shown in Fig. 3. The patch  $M_1$  consists of the four smallest elements of size  $\sigma^n$ . On a patch  $M_j$ ,  $j \geq 2$ , the four inner faces will have to be treated separately. We denote them by  $f_{11}^j$ ,  $f_{21}^j$ ,  $f_{23}^j$  and  $f_{33}^j$ , as illustrated in Fig. 3.

For  $2 \leq j \leq n$ , we introduce the seminorm

$$|q|_{h,j}^2 = \sum_{i=1,2} h_{f_{i1}^j} \int_{f_{i1}^j} |\llbracket q \rrbracket|^2 ds + \sum_{i=2,3} h_{f_{i3}^j} \int_{f_{i3}^j} |\llbracket q \rrbracket|^2 ds.$$

We then set

$$|q|_h^2 = \sum_{j=2}^n |q|_{h,j}^2. \quad (8.2)$$

First, we prove the following technical result.

**LEMMA 30** Let  $\mathcal{T}_e^{n,\sigma}$  be an edge patch on  $\widehat{Q}$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Then there exists a constant that solely depends on  $\sigma$ , but is independent of  $k$ ,  $n$ , and the aspect ratio of the elements in  $\mathcal{T}_e^{n,\sigma}$ , such that

$$\left| \int_{\mathcal{E}_T \cap \widehat{Q}} \llbracket q \rrbracket \cdot \{\{\mathbf{u} - \Pi^k \mathbf{u}\}\} ds \right| \leq C |\mathbf{u}|_{1,\widehat{Q}} |q|_h,$$

for  $\mathbf{u} \in H^1(\widehat{Q})^3$ ,  $q \in Q_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$ , and  $\Pi^k \mathbf{u}$  the interpolant in (8.1).

*Proof.* By density, we may assume that  $\mathbf{u} \in C^\infty(\overline{\widehat{Q}})^3$ . We note that the integral over  $\mathcal{E}_T \cap \widehat{Q}$  can be written as a sum of contributions over faces  $f \subset \mathcal{E}_T$ . In addition, if  $f$  is a regular face, i.e. it is an entire face of two neighbouring elements  $K$  and  $K'$ , then the second orthogonality condition (7.4) ensures that its contribution vanishes. Indeed, in this case  $\mathbf{u}$  and  $\Pi^k \mathbf{u}$  have a continuous normal component across  $f$  and the normal vector  $\llbracket q \rrbracket$  belongs to  $\mathbb{Q}_{k,k}(f)$ . Therefore, we obtain

$$\begin{aligned} \int_{\mathcal{E}_T \cap \widehat{Q}} \llbracket q \rrbracket \cdot \{\mathbf{u} - \Pi^k \mathbf{u}\} \, ds &= \sum_{j=2}^n \sum_{i=1,2} \int_{f_{i1}^j} \llbracket q \rrbracket \cdot \{\mathbf{u} - \Pi^k \mathbf{u}\} \, ds \\ &\quad + \sum_{j=2}^n \sum_{i=2,3} \int_{f_{i3}^j} \llbracket q \rrbracket \cdot \{\mathbf{u} - \Pi^k \mathbf{u}\} \, ds. \end{aligned}$$

We first bound the contribution over  $f = f_{11}^j$ . Denote by  $K_1$  and  $K_2$  the elements that share  $f$ , assuming that  $f$  is an entire face of  $K_1$ . Let  $q_1$  and  $q_2$  be the restrictions of  $q$  to  $K_1$  and  $K_2$ , respectively. Further, we set  $\mathbf{v} = \Pi^k \mathbf{u}$ , as well as  $\mathbf{u}|_{K_i} = \mathbf{u}^i = (u_x^i, u_y^i, u_z^i)$  and  $\mathbf{v}^i = (v_x^i, v_y^i, v_z^i)$  for  $i = 1, 2$ . Therefore,

$$\begin{aligned} \int_f \llbracket q \rrbracket \cdot \{\mathbf{u} - \Pi^k \mathbf{u}\} \, ds &= \frac{1}{2} \int_f (q_1 - q_2)(u_x^1 - v_x^1) \, ds \\ &\quad + \frac{1}{2} \int_f (q_1 - q_2)(u_x^2 - v_x^2) \, ds \\ &= \frac{1}{2} T_1 + \frac{1}{2} T_2. \end{aligned}$$

We start with a bound for  $T_1$  and proceed as in the proof of Lemma 25. We use the representation result of Lemma 22, the fact that  $(q_1 - q_2)$  is a polynomial of degree  $k$  in the  $z$ -direction, the properties of  $\pi_k^{0,z}$  and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} |T_1| &= \left| \int_f (q_1 - q_2)(u_x^1 - \pi_k^{0,z} \pi_{k+1}^{1,x} \pi_k^{0,y} u_x^1) \, ds \right| \\ &= \left| \int_f (q_1 - q_2)(u_x^1 - \pi_{k+1}^{1,x} \pi_k^{0,y} u_x^1) \, ds \right| \\ &\leq (h_f \int_f |\llbracket q \rrbracket|^2 \, ds)^{\frac{1}{2}} (h_f^{-1} \int_f (u_x^1 - \pi_{k+1}^{1,x} \pi_k^{0,y} u_x^1)^2 \, ds)^{\frac{1}{2}}. \end{aligned}$$

Since  $\pi_{k+1}^{1,x} \pi_k^{0,y}$  is the first component of the two-dimensional Raviart–Thomas projector and since the underlying two-dimensional geometric mesh  $\mathcal{T}_{xy}^{n,\sigma}$  is shape-regular, we can apply Lemma 18 and obtain

$$h_f^{-1} \int_f (u_x^1 - \pi_{k+1}^{1,x} \pi_k^{0,y} u_x^1)^2 \, ds \leq C \|\partial_x \mathbf{u}^1\|_{0,K_1}^2 + C \|\partial_y \mathbf{u}^1\|_{0,K_1}^2.$$

Combining with the analogous argument for  $T_2$  gives

$$\begin{aligned} \left| \int_f \llbracket q \rrbracket \cdot \{\mathbf{u} - \Pi^k \mathbf{u}\} \, ds \right| &\leq C (h_f \int_f |\llbracket q \rrbracket|^2 \, ds)^{\frac{1}{2}} \\ &\quad \cdot (\|\partial_x \mathbf{u}^1\|_{0,K_1}^2 + \|\partial_x \mathbf{u}^1\|_{0,K_1}^2 + \|\partial_x \mathbf{u}^2\|_{0,K_2}^2 + \|\partial_x \mathbf{u}^2\|_{0,K_2}^2)^{\frac{1}{2}}. \end{aligned}$$

The contributions of the other faces  $f_{ik}^j$  can be bounded analogously. Summing over all faces and using the Cauchy–Schwarz inequality completes the proof.  $\square$

The previous lemma allows us to prove a stability result that is weaker than the inf-sup condition in Theorem 27.

LEMMA 31 Let  $\mathcal{T}_e^{n,\sigma}$  be an edge patch on  $\widehat{Q}$  with grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. Then, for  $k \geq 1$ ,

$$\sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})} \frac{B_{h,\widehat{Q}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,\widehat{Q}}} \geq Ck^{-1} \|q\|_{0,\widehat{Q}} \left( 1 - \frac{|q|_h}{\|q\|_{0,\widehat{Q}}} \right), \quad q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q}),$$

with a constant  $C > 0$  that solely depends on  $\sigma$  and  $\delta_0$ , but is independent of  $k$ ,  $n$ , and the aspect ratio of the elements in  $\mathcal{T}_e^{n,\sigma}$ .

*Proof.* Let  $q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$ . Thanks to the continuous inf-sup condition (2.3) for  $\Omega = \widehat{Q}$ , there exists  $\mathbf{u} \in H_0^1(\widehat{Q})^3$  such that

$$B(\mathbf{u}, q) = \|q\|_{0,\widehat{Q}}^2, \quad |\mathbf{u}|_{1,\widehat{Q}} \leq (1/\gamma_{\widehat{Q}}) \|q\|_{0,\widehat{Q}}. \quad (8.3)$$

We choose  $\mathbf{v} = \Pi^{k-1}\mathbf{u}$ , with  $\Pi^{k-1}$  the interpolant in (8.1). We then have

$$B_{h,\widehat{Q}}(\mathbf{v}, q) = B(\mathbf{u}, q) - B_{h,\widehat{Q}}(\mathbf{u} - \Pi^{k-1}\mathbf{u}, q) \geq \|q\|_{0,\widehat{Q}}^2 - |B_{h,\widehat{Q}}(\mathbf{u} - \Pi^{k-1}\mathbf{u}, q)|.$$

Using (6.4) and the first orthogonality property in (7.4), we can write

$$\begin{aligned} B_{h,\widehat{Q}}(\mathbf{u} - \Pi^{k-1}\mathbf{u}, q) &= \int_{\widehat{Q}} (\mathbf{v} - \Pi^{k-1}\mathbf{u}) \cdot \nabla_h q \, d\mathbf{x} \\ &\quad - \int_{\mathcal{E}_T \cap \widehat{Q}} \llbracket q \rrbracket \cdot \{\{\mathbf{u} - \Pi^{k-1}\mathbf{u}\}\} \, ds \\ &= - \int_{\mathcal{E}_T \cap \widehat{Q}} \llbracket q \rrbracket \cdot \{\{\mathbf{u} - \Pi^{k-1}\mathbf{u}\}\} \, ds. \end{aligned}$$

Using Lemma 30 and the second bound of (8.3) thus yields

$$B_h(\mathbf{v}, q) = B_h(\mathbf{u}, q) + B_h(\mathbf{v} - \mathbf{u}, q) \geq \|q\|_{0,\widehat{Q}}^2 - C \|q\|_{0,\widehat{Q}} |q|_h. \quad (8.4)$$

Using Corollary 29 and (8.3) gives

$$\|\mathbf{v}\|_{h,\widehat{Q}} \leq Ck |\mathbf{u}|_{1,\widehat{Q}} \leq Ck \|q\|_{0,\widehat{Q}},$$

which concludes the proof.  $\square$

We end this section by providing a second inf-sup condition in terms of the pressure seminorm  $|\cdot|_h$  in (8.2). Its proof is given in Appendix A.

LEMMA 32 Let  $\mathcal{T}_e^{n,\sigma}$  be an edge patch on  $\widehat{Q}$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. For  $k \geq 1$ ,

$$\sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})} \frac{B_{h,\widehat{Q}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,\widehat{Q}}} \geq Ck^{-3/2} |q|_h, \quad q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q}),$$

with a constant  $C > 0$  that solely depends on  $\sigma$  and  $\delta_0$ , but is independent of  $k$ ,  $n$ , and the aspect ratio of the elements in  $\mathcal{T}_e^{n,\sigma}$ .

### 8.3 Proof of Theorem 27

We now combine Lemmas 31 and 32. If  $t$  denotes the ratio  $|q|_h/\|q\|_{0,\widehat{Q}}$ , we find

$$\sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})} \frac{B_{h,\widehat{Q}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,\widehat{Q}}} \geq C k^{-3/2} \|q\|_{0,\widehat{Q}} \min_{t \geq 0} f(t), \quad q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q}),$$

where  $f(t) = \max\{1-t, t\}$ . The proof is concluded by noting that  $\min_{t \geq 0} f(t)$  is equal to  $1/2$ .

## 9. Divergence stability on geometric edge meshes

In this section, we consider geometric edge meshes on  $\Omega$  and prove Theorem 9.

### 9.1 Trivial patch

We have the following result.

**THEOREM 33** Let  $\widehat{\mathcal{T}}$  be the trivial patch given by the mesh  $\widehat{\mathcal{T}} = \{\widehat{Q}\}$ . For  $k \geq 1$ ,

$$\sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\widehat{\mathcal{T}}; \widehat{Q})} \frac{B_{h,\widehat{Q}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,\widehat{Q}}} \geq C k^{-1} \|q\|_{0,\widehat{Q}}, \quad q \in \mathcal{Q}_h^{k-1}(\widehat{\mathcal{T}}; \widehat{Q}),$$

with a constant  $C > 0$  independent of  $k$ .

*Proof.* Since  $\widehat{\mathcal{T}}$  only consists of one element, given  $\mathbf{u} \in H_0^1(\widehat{Q})^3$ , we have

$$B_{h,\widehat{Q}}(\Pi_{\widehat{Q}}^{k-1} \mathbf{u}, q) = B(\mathbf{u}, q), \quad \|\Pi_{\widehat{Q}}^{k-1} \mathbf{u}\|_{h,\widehat{Q}} \leq C k |\mathbf{u}|_{1,\widehat{Q}},$$

for all  $q \in \mathcal{Q}_h^{k-1}(\widehat{\mathcal{T}}; \widehat{Q})$ , where  $\Pi_{\widehat{Q}}^{k-1}$  is the Raviart–Thomas interpolant from Section 7.3 on  $\widehat{Q}$  and we have used the orthogonality properties in (7.4) and the results in Schötzau et al. (2003, Lemmas 6.9 and 6.10). We note that  $\Pi_{\widehat{Q}}^{k-1} \mathbf{u} \in \widetilde{\mathbf{V}}_h^k(\widehat{\mathcal{T}}; \widehat{Q})$ . The divergence stability property is then a consequence of the continuous inf-sup condition (2.3) for  $\Omega = \widehat{Q}$ .  $\square$

### 9.2 Corner patches

The stability of corner patches is proven by using the macro-element technique.

**THEOREM 34** Let  $\mathcal{T}_c^{n,\sigma}$  be a corner patch on  $\widehat{Q}$  with a grading factor  $\sigma \in (0, 1)$  and  $n$  layers of refinement. For  $k \geq 2$ ,

$$\sup_{0 \neq \mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_c^{n,\sigma}; \widehat{Q})} \frac{B_{h,\widehat{Q}}(\mathbf{v}, q)}{\|\mathbf{v}\|_{h,\widehat{Q}}} \geq C k^{-3/2} \|q\|_{0,\widehat{Q}}, \quad q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_c^{n,\sigma}; \widehat{Q}),$$

with a constant  $C > 0$  that solely depends on  $\sigma$  and  $\delta_0$ , but is independent of  $k$ ,  $n$ , and the aspect ratio of the elements in  $\mathcal{T}_c^{n,\sigma}$ .

*Proof.* We use the macro-element technique in Theorem 12 and Proposition 13 with  $\Omega = \widehat{Q}$ , the corner mesh  $\mathcal{T} = \mathcal{T}_c^{n,\sigma}$  and the macro-element mesh  $\mathcal{T}_m = \mathcal{T}_{c,m}^{n,\sigma}$ . The stability result (6.7) for piecewise constant pressures on  $\mathcal{T}_m$  then trivially holds by choosing  $\mathbf{X}_h$  as the space of continuous, piecewise quadratic velocities; see Stenberg & Suri (1996) for regular meshes and Toselli & Schwab (2003) for irregular meshes. Condition (6.15) in Proposition 13 is satisfied due to Theorem 33 (trivial patch) and by noting that the anisotropically refined elements in  $\mathcal{T}_{c,m}^{n,\sigma}$  are particular edge patches that are stable according to Theorem 27.  $\square$

### 9.3 Proof of Theorem 9

The proof of Theorem 9 now follows similarly from the macro-element technique in Theorem 12 and Proposition 13. Indeed, the low-order stability result (6.7) on  $\mathcal{T}_m$  holds by choosing  $\mathbf{X}_h$  again as the space of continuous, piecewise quadratic velocities; see Stenberg & Suri (1996). Condition (6.15) in Proposition 13 is satisfied due to Theorem 33 (trivial patch), Theorem 27 (edge patch) and Theorem 34 (corner patch).

REMARK 35 Since we choose the low-order space  $\mathbf{X}_h$  in (6.7) as the space of continuous, piecewise quadratic velocities, Theorems 9 and 34 only hold for  $k \geq 2$ .

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### REFERENCES

- AINSWORTH, M. & COGGINS, P. (2000) The stability of mixed  $hp$ -finite element methods for Stokes flow on high aspect ratio elements. *SIAM J. Numer. Anal.*, **38**, 1721–1761.
- AINSWORTH, M. & PINCHEDEZ, K. (2002)  $hp$ -Approximation theory for BDFM and RT finite elements on quadrilaterals. *SIAM J. Numer. Anal.*, **40**, 2047–2068.
- ANDERSSON, B., FALK, U., BABUŠKA, I. & VON PETERSDORFF, T. (1995) Reliable stress and fracture mechanics analysis of complex aircraft components using a  $hp$ -version FEM. *Int. J. Numer. Methods Engrg.*, **38**, 2135–2163.
- ARNOLD, D., BREZZI, F., COCKBURN, B. & MARINI, L. (2001) Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, **39**, 1749–1779.
- BABUŠKA, I. & GUO, B. (1996) Approximation properties of the  $hp$ -version of the finite element method. *Comput. Methods Appl. Mech. Engrg.*, **133**, 319–346.
- BAKER, G., JUREIDINI, W. & KARAKASHIAN, O. (1990) Piecewise solenoidal vector fields and the Stokes problem. *SIAM J. Numer. Anal.*, **27**, 1466–1485.
- BERNARDI, C. & MADAY, Y. (1997) Spectral methods. *Handbook of Numerical Analysis*, Vol. 5. Amsterdam: North-Holland, pp. 209–485.
- BERNARDI, C. & MADAY, Y. (1999) Uniform inf-sup conditions for the spectral discretization of the Stokes problem. *Math. Models Methods Appl. Sci.*, **9**, 395–414.
- BREZZI, F. & FALK, R. (1991) Stability of higher-order Hood–Taylor methods. *SIAM J. Numer. Anal.*, **28**, 581–590.
- BREZZI, F. & FORTIN, M. (1991) Mixed and hybrid finite element methods. *Springer Series in Computational Mathematics*, Vol. 15. New York: Springer.

- CANUTO, C. & QUARTERONI, A. (1982) Approximation results for orthogonal polynomials in Sobolev spaces. *Math. Comput.*, **38**, 67–86.
- CHILTON, L. & SURI, M. (2000) On the construction of stable curvilinear  $p$ -version elements for mixed formulations of elasticity and Stokes flow. *Numer. Math.*, **86**, 611–648.
- COCKBURN, B. (1999) Discontinuous Galerkin methods for convection-dominated problems. *High-Order Methods for Computational Physics*, Vol. 9. (T. Barth & H. Deconink, eds). New York: Springer, pp. 69–224.
- COCKBURN, B. & SHU, C.-W. (2001) Runge–Kutta discontinuous Galerkin methods for convection-dominated problems. *J. Sci. Comput.*, **16**, 173–261.
- COCKBURN, B., KANSCHAT, G. & SCHÖTZAU, D. (2003) The local discontinuous Galerkin method for the Oseen equations. *Math. Comput.*, published electronically.
- COCKBURN, B., KANSCHAT, G., SCHÖTZAU, D. & SCHWAB, C. (2002) Local discontinuous Galerkin methods for the Stokes system. *SIAM J. Numer. Anal.*, **40**, 319–343.
- COCKBURN, B., KARNIADAKIS, G. & SHU, C.-W. (eds) (2000) Discontinuous Galerkin Methods. *Theory, Computation and Applications*, Lecture Notes in Computer Science Engineering, Vol. 11. New York: Springer.
- FRANCA, L. & STENBERG, R. (1991) Error analysis of some Galerkin-least-squares methods for the elasticity equations. *SIAM J. Numer. Anal.*, **28**, 1680–1697.
- GEORGIOULIS, E. & SÜLI, E. (2001) *hp-DGFEM on shape-irregular meshes: reaction–diffusion*. Technical Report NA 01–09. Oxford University Computing Laboratory.
- GERDES, K., MELENK, J., SCHWAB, C. & SCHÖTZAU, D. (2001) The  $hp$ -version of the streamline diffusion finite element method in two space dimensions. *Math. Models Methods Appl. Sci.*, **11**, 301–337.
- GIRAULT, V. & RAVIART, P. (1986) *Finite Element Methods for Navier–Stokes Equations*. New York: Springer.
- GIRAULT, V., RIVIÈRE, B. & WHEELER, M. (2002) A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier–Stokes problems. *Technical Report 02-08*. Austin: TICAM, UT.
- HANSBO, P. & LARSON, M. (2002) Discontinuous finite element methods for incompressible and nearly incompressible elasticity by use of Nitsche’s method. *Comput. Methods Appl. Mech. Engrg.*, **191**, 1895–1908.
- HIENTZSCH, B. (2001) Fast solvers and domain decomposition preconditioners for spectral element discretizations of problems in  $H(\text{curl})$ . *Ph.D. Thesis*, Courant Institute of Mathematical Sciences, Technical Report 823, Department of Computer Science, Courant Institute of Mathematical Sciences, New York University.
- HOUSTON, P., SCHWAB, C. & SÜLI, E. (2002) Discontinuous  $hp$ -finite element methods for advection–diffusion–reaction problems. *SIAM J. Numer. Anal.*, **39**, 2133–2163.
- KARAKASHIAN, O. & JUREIDINI, W. (1998) A nonconforming finite element method for the stationary Navier–Stokes equations. *SIAM J. Numer. Anal.*, **35**, 93–120.
- LASSER, C. & TOSELLI, A. (2003) An overlapping domain decomposition preconditioner for a class of discontinuous Galerkin approximations of advection–diffusion problems. *Math. Comput.*, **72**, 1215–1238.
- MELENK, J. & SCHWAB, C. (1998)  $hp$ -FEM for reaction–diffusion equations, I. Robust exponential convergence. *SIAM J. Numer. Anal.*, **35**, 1520–1557.
- SCHÖTZAU, D. & SCHWAB, C. (1998) Mixed  $hp$ -FEM on anisotropic meshes. *Math. Models Methods Appl. Sci.*, **8**, 787–820.
- SCHÖTZAU, D. & WIHLE, T. (2002) Exponential convergence of mixed  $hp$ -DGFEM for Stokes flow in polygons. *Technical Report 02-15* Department of Mathematics, University of Basel, to



appear in Numer. Math.

- SCHÖTZAU, D., SCHWAB, C. & STENBERG, R. (1999) Mixed  $hp$ -FEM on anisotropic meshes, II. Hanging nodes and tensor products of boundary layer meshes. *Numer. Math.*, **83**, 667–697.
- SCHÖTZAU, D., SCHWAB, C. & TOSELLI, A. (2003) Mixed  $hp$ -DGFEM for incompressible flows. *SIAM J. Numer. Anal.*, **40**, 2171–2194.
- SCHWAB, C. (1998) *p- and hp-FEM—Theory and Application to Solid and Fluid Mechanics*. Oxford: Oxford University Press.
- SCHWAB, C. & SURI, M. (1996) The  $p$ - and  $hp$ -version of the finite element method for problems with boundary layers. *Math. Comput.*, **65**, 1403–1429.
- SCHWAB, C., SURI, M. & XENOPHONTOS, C. (1998) The  $hp$ -finite element method for problems in mechanics with boundary layers. *Comput. Methods Appl. Mech. Engrg.*, **157**, 311–333.
- STENBERG, R. (1990) Error analysis of some finite element methods for Stokes problem. *Math. Comput.*, **54**, 495–508.
- STENBERG, R. & SURI, M. (1996) Mixed  $hp$ -finite element methods for problems in elasticity and Stokes flow. *Numer. Math.*, **72**, 367–389.
- TOSELLI, A. (2002)  $hp$ -Discontinuous Galerkin approximations for the Stokes problem. *Math. Models Methods Appl. Sci.*, **12**, 1565–1616.
- TOSELLI, A. & SCHWAB, C. (2003) Mixed  $hp$ -finite element approximations on geometric edge and boundary layer meshes in three dimensions. *Numer. Math.*, **94**, 771–801.

## Appendix. Proof of Lemma 32

We proceed in several steps.

*Step 1: A lifting operator.* Let  $K = K_{xyz} = I_x \times I_y \times I_z$  with  $I_x = (x_1, x_2)$  and  $h_x = x_2 - x_1$ . Consider the face  $f_{x_1} = \{x = x_1\}$ . We define the operator  $\mathcal{E}_{k,K}^{f_{x_1}} : \mathbb{Q}_{k,k}(f_{x_1}) \rightarrow \mathbb{Q}_{k+1,k,k}(K)$  by

$$(\mathcal{E}_{k,K}^{f_{x_1}} \varphi)(x, y, z) = M_k^{f_{x_1}}(x) \varphi(y, z), \quad M_k^{f_{x_1}}(x) = \frac{(-1)^{k+1}}{2} (L_{k+1}(x) - L_k(x)),$$

where  $\{L_i\}$  denote the Legendre polynomials on  $I_x$ . This lifting operator was originally proposed in Ainsworth & Pinchedez (2002) and then employed in Schötzau *et al.* (2003). Note that  $(\mathcal{E}_{k,K}^{f_{x_1}} \varphi)(x_1, y, z) = \varphi(y, z)$  and  $(\mathcal{E}_{k,K}^{f_{x_1}} \varphi)(x_2, y, z) = 0$ , thanks to the properties of  $\{L_i\}$ , cf. Bernardi & Maday (1997, Section 3). From the results in Schötzau *et al.* (2003, Lemma 6.8) and a scaling argument we have

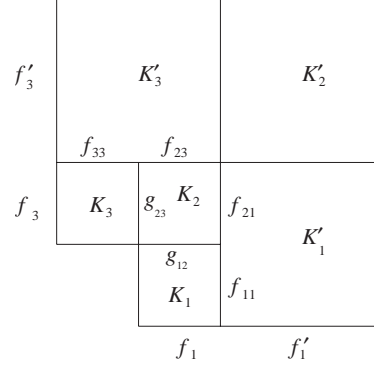
$$\|M_{k,K}^{f_{x_1}}\|_{0,I_x}^2 \leq Ch_x k^{-1}, \quad |M_{k,K}^{f_{x_1}}|_{1,I_x}^2 \leq Ch_x^{-1} k^3. \quad (\text{A.1})$$

Analogous definitions and bounds hold for the other faces of  $K$ . Furthermore, for  $\varphi \in \mathbb{Q}_{k,k}(f_{x_1})$ , we have

$$\int_K (\mathcal{E}_{k,K}^{f_{x_1}} \varphi) w \, d\mathbf{x} = 0 \quad \forall w \in \mathbb{Q}^{k-1,k,k}(K). \quad (\text{A.2})$$

This follows from the definition of the lifting operators and orthogonality properties of the Legendre polynomials. Analogous results are valid for the other faces.

*Step 2: Stability on the layer  $j$ .* Let  $M_j$ ,  $2 \leq j \leq n$ , denote the patch of elements illustrated in Fig. 3. It consists of six elements: we denote the inner elements by  $K_i$ ,  $i =$

FIG. A.1. Two-dimensional illustration of the elements and faces in a patch  $M_j$ , for  $\sigma = 0.5$ .

1, 2, 3, and the outer ones by  $K'_i$ ,  $i = 1, 2, 3$ . The four interior faces connecting elements  $\{K_i\}$  and  $\{K'_i\}$  are denoted by  $f_{11}$ ,  $f_{21}$ ,  $f_{23}$ , and  $f_{33}$ . These faces are entire faces of the inner elements only. The faces connecting the inner elements are  $g_{12}$  and  $g_{23}$ . The exterior faces are denoted by  $f_1$ ,  $f'_1$  and  $f_3$ ,  $f'_3$ , respectively. In Fig. A.1, we show the configuration of the elements and faces in  $M_j$ .

Let  $q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$  for  $k \geq 1$ . We denote  $q|_{K_i}$  by  $q_i$  and  $q|_{K'_i}$  by  $q'_i$ ,  $i = 1, 2, 3$ . Using the lifting operators from Step 1, we define the function  $\mathbf{v} \in \mathbf{V}_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$  by

$$\begin{aligned} \mathbf{v}|_{K_1} &= \mathbf{v}^1 = (-h_{f_{11}} \mathcal{E}_{k-1,K_1}^{f_{11}}(q_1 - q'_1), 0, 0), \\ \mathbf{v}|_{K_2} &= \mathbf{v}^2 = (-h_{f_{21}} \mathcal{E}_{k-1,K_2}^{f_{21}}(q_2 - q'_1), -h_{f_{23}} \mathcal{E}_{k-1,K_2}^{f_{23}}(q_2 - q'_3), 0), \\ \mathbf{v}|_{K_3} &= \mathbf{v}^3 = (0, -h_{f_{33}} \mathcal{E}_{k-1,K_3}^{f_{33}}(q_3 - q'_3), 0), \end{aligned}$$

and by  $\mathbf{v}|_K = \mathbf{0}$  on the remaining elements of  $\mathcal{T}_e$ . In particular, note that the function  $\mathbf{v}$  is equal to zero on the faces adjacent to layer  $j+1$  and layer  $j-1$  and satisfies  $\mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$ .

We further note that  $\int_{K_i} \nabla q \cdot \mathbf{v} \, d\mathbf{x} = 0$ ,  $i = 1, 2, 3$ . This follows from the definition of  $\mathbf{v}$  and property (A.2). We define  $B_{h,M_j}(\cdot, \cdot)$  and  $\|\cdot\|_{0,M_j}$  as in (6.1) and (6.2), respectively. Thus,

$$\begin{aligned} B_{h,\widehat{Q}}(\mathbf{v}, q) &= B_{h,M_j}(\mathbf{v}, q) = - \int_{\mathcal{E}_{\mathcal{T}} \cap M_j} \llbracket q \rrbracket \cdot \{\{\mathbf{v}\}\} \, ds \\ &= \frac{1}{2} \sum_{i=1,2} \int_{f_{i1}} h_{f_{i1}} |\llbracket q \rrbracket|^2 \, ds + \frac{1}{2} \sum_{i=2,3} \int_{f_{i3}} h_{f_{i3}} |\llbracket q \rrbracket|^2 \, ds = \frac{1}{2} |q|_{h,j}^2. \end{aligned} \quad (\text{A.3})$$

Next, we bound the norm  $\|\mathbf{v}\|_{h,M_j}$  in terms of  $|q|_{h,j}$ .

We start by considering the element  $K_1$ . Writing  $K_1 = I_x \times I_y \times (-1, 1)$ , we have

$$\|\partial_x \mathbf{v}^1\|_{0,K_1}^2 = h_{f_{11}}^2 |M_{k-1}^{f_{11}}|_{1,I_x}^2 \int_{f_{11}} |\llbracket q \rrbracket|^2 \, ds \leq Ch_{f_{11}} k^3 \int_{f_{11}} |\llbracket q \rrbracket|^2 \, ds.$$

Here, we used the second estimate in (A.1) and the fact that all mesh sizes are comparable in the underlying two-dimensional mesh  $\mathcal{T}_{xy}^{n,\sigma}$ . Then, from the inverse estimate for polynomials in Schwab (1998, Theorem 3.91) and the first estimate in (A.1), we have

$$\begin{aligned} \|\partial_y \mathbf{v}^1\|_{0,K_1}^2 &= h_{f_{11}}^2 \|M_{k-1}^{f_{11}}\|_{0,I_x}^2 \int_{f_{11}} |\partial_y \llbracket q \rrbracket|^2 ds \\ &\leq Ch_{f_{11}}^3 k^{-1} h_{f_{11}}^{-2} k^4 \int_{f_{11}} |\llbracket q \rrbracket|^2 ds = Ch_{f_{11}} k^3 \int_{f_{11}} |\llbracket q \rrbracket|^2 ds. \end{aligned}$$

Similarly,

$$\begin{aligned} \|\partial_z \mathbf{v}^1\|_{0,K_1}^2 &= h_{f_{11}}^2 \|M_{k-1}^{f_{11}}\|_{0,I_x}^2 \int_{f_{11}} |\partial_z \llbracket q \rrbracket|^2 ds \\ &\leq Ch_{f_{11}}^3 k^{-1} k^4 \int_{f_{11}} |\llbracket q \rrbracket|^2 ds = Ch_{f_{11}} k^3 \int_{f_{11}} |\llbracket q \rrbracket|^2 ds. \end{aligned}$$

Again, we used (A.1) and the inverse estimate in Schwab (1998, Theorem 3.91) on the interval  $(-1, 1)$  in the  $z$ -direction.

The same techniques yield the analogous estimates for  $\mathbf{v}$  on the elements  $K_2$  and  $K_3$ . It remains to bound the jumps of  $\mathbf{v}$  over the various faces.

We start by considering the jump over  $f_{11}$ . Thanks to (3.1), we have

$$\int_{f_{11}} \delta |\llbracket \mathbf{v} \rrbracket|^2 ds \leq Ck^2 h_{f_{11}}^{-1} \int_{f_{11}} h_{f_{11}}^2 |\llbracket q \rrbracket|^2 ds = Ch_{f_{11}}^2 k^2 \int_{f_{11}} |\llbracket q \rrbracket|^2 ds.$$

The jump over  $f_{33}$  can be bounded similarly. Let us now consider the face  $g_{12}$ . Writing  $g_{12} = I_x \times \{y_1\} \times (-1, 1)$ , we have

$$\begin{aligned} \int_{g_{12}} \delta |\llbracket \mathbf{v} \rrbracket|^2 ds &\leq k^2 h_{g_{12}}^{-1} C \int_{g_{12}} h_{f_{11}}^2 |\mathcal{E}_{k-1,K_1}^{f_{11}}(q_1 - q'_1)|^2 ds \\ &\quad + k^2 h_{g_{12}}^{-1} C \int_{g_{12}} h_{f_{21}}^2 |\mathcal{E}_{k-1,K_2}^{f_{21}}(q_2 - q'_1)|^2 ds \\ &\leq Ck^2 h_{f_{11}} \|M_{k-1}^{f_{11}}\|_{0,I_x}^2 \int_{-1}^1 |\llbracket q \rrbracket|_{f_{11}}(y_1, z)|^2 dz \\ &\quad + Ck^2 h_{f_{21}} \|M_{k-1}^{f_{21}}\|_{0,I_x}^2 \int_{-1}^1 |\llbracket q \rrbracket|_{f_{21}}(y_1, z)|^2 dz \\ &\leq Ckh_{f_{11}}^2 \int_{-1}^1 |\llbracket q \rrbracket|_{f_{11}}(y_1, z)|^2 dz + Ckh_{f_{21}}^2 \int_{-1}^1 |\llbracket q \rrbracket|_{f_{21}}(y_1, z)|^2 dz \\ &\leq Ck^3 h_{f_{11}} \int_{f_{11}} |\llbracket q \rrbracket|^2 ds + Ck^3 h_{f_{21}} \int_{f_{21}} |\llbracket q \rrbracket|^2 ds. \end{aligned}$$

Here, we used the definition of  $\mathbf{v}$ , the fact that all mesh sizes are comparable in the underlying two-dimensional mesh  $\mathcal{T}_{xy}^{n,\sigma}$ , the  $L^2$ -bound in (A.1), and the inverse estimate in Schwab (1998, Theorem 3.91) for polynomials.

Exactly the same techniques allow us to bound the jumps over  $g_{23}$ ,  $f_{23}$ ,  $f_{21}$ ,  $f_1$  and  $f_3$  in terms of  $|q|_{h,j}$ . Finally, the same approach gives bounds for the top and bottom faces  $z = \pm 1$ .

Combining the above estimates yields

$$\|\mathbf{v}\|_{h,\widehat{Q}}^2 = \|\mathbf{v}\|_{h,M_j}^2 \leq Ck^3|q|_{h,j}^2. \quad (\text{A.4})$$

*Step 3: The assertion.* Let  $q \in \mathcal{Q}_h^{k-1}(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$ . On  $M_j$ , there is a velocity field  $\mathbf{v}_j$  that satisfies (A.3) and (A.4). We set  $\mathbf{v} = \sum_{j=2}^n \mathbf{v}_j$ . By construction,  $\mathbf{v} \in \widetilde{\mathbf{V}}_h^k(\mathcal{T}_e^{n,\sigma}; \widehat{Q})$ . Using (A.3), we find

$$B_{h,\widehat{Q}}(\mathbf{v}, q) = \sum_{j=2}^n B_{h,\widehat{Q}}(\mathbf{v}_j, q) = \sum_{j=2}^n B_{h,M_j}(\mathbf{v}_j, q) \geq C \sum_{j=2}^m |q|_{h,j}^2 = C|q|_h^2.$$

Furthermore, from (A.4) and the fact that the support of the fields  $\mathbf{v}_j$  is locally in the patch  $M_j$ , we have  $\|\mathbf{v}\|_{h,\widehat{Q}}^2 \leq C|q|_h^2$ . This concludes the proof.  $\square$